

NOTES ON THE PARITY CONJECTURE

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The main purpose of these notes is to prove, in a reasonably self-contained way, that finiteness of the Tate-Shafarevich group implies the parity conjecture for elliptic curves over number fields. Along the way, we review local and global root numbers of elliptic curves and their classification, and we end by discussing some peculiar consequences of the parity conjecture.

Essentially nothing here is new, and the notes follow closely the papers [11]–[17], all joint with Vladimir Dokchitser. There are only some additional shortcuts replacing a few technical computations of root numbers and Tamagawa numbers by a ‘deforming to totally real fields’ argument. Also, this is not meant to be a complete survey of results on the parity conjecture, and many important results, especially those concerning Selmer groups, are not mentioned.

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1. BIRCH-SWINNERTON-DYER AND PARITY

1.1. Conjectures and the main result. Throughout the notes K denotes a number field. Suppose E/K is an elliptic curve, say in Weierstrass form,

$$y^2 = x^3 + ax + b, \quad a, b \in K.$$

The set $E(K)$ of K -rational solutions (x, y) to this equation together with a point at infinity O forms an abelian group, which is finitely generated by famous theorems of Mordell and Weil. The primary arithmetic invariant of E/K is its rank:

Definition. The *Mordell-Weil rank* $\text{rk } E/K$ is the \mathbb{Z} -rank of $E(K)/\text{torsion}$.

The celebrated conjecture of Birch and Swinnerton-Dyer relates it to another fundamental invariant, the *L-function* $L(E/K, s)$. We define it in §3, together with its *conductor* $N \in \mathbb{Z}$ and the *global root number* $w(E/K) = \pm 1$. For now it suffices to say that $L(E/K, s)$ is given by a Dirichlet series,

$$L(E/K, s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

with $a_n \in \mathbb{Z}$, and it converges for $\text{Re } s > 3/2$.

Hasse-Weil Conjecture. *The function $L(E/K, s)$ has an analytic continuation to the whole of \mathbb{C} . The completed L-function*

$$L^*(E/K, s) = \Gamma\left(\frac{s}{2}\right)^d \Gamma\left(\frac{s+1}{2}\right)^d \pi^{-ds} N^{s/2} L(E/K, s) \quad (d = [K : \mathbb{Q}])$$

satisfies a functional equation

$$L^*(E/K, 2-s) = w(E/K) L^*(E/K, s).$$

Granting the analytic continuation, we may state

Birch-Swinnerton-Dyer Conjecture I.

$$\text{ord}_{s=1} L(E/K, s) = \text{rk } E/K.$$

This remarkable conjecture relates arithmetic properties of E to analytic properties of its L -function. It was originally stated over the rationals, extended to all abelian varieties over global fields by Tate and vastly generalised by Deligne, Gross, Beilinson, Bloch and Kato. One immediate consequence of the two conjectures above is

Parity Conjecture. $(-1)^{\text{rk } E/K} = w(E/K)$.

In particular, an elliptic curve whose root number is -1 must have infinitely many rational points. This is a purely arithmetic statement which does not involve L -functions and it might appear to be simpler than the two conjectures above. However, it is remarkably hard, and we have no approach to resolve it in any kind of generality. It has several important consequences: for instance, it settles Hilbert's 10th problem over rings of

integers of arbitrary number fields [36] and implies most remaining cases of the congruent number problem.

The difficulty is that we know virtually nothing about the rank of a general elliptic curve, as it is very hard to distinguish rational points from the elements of III :

Definition. The *Tate-Shafarevich group* of E/K is defined by

$$\text{III}_{E/K} = \ker(H^1(K, E(\bar{K})) \rightarrow \prod_v H^1(K_v, E(\bar{K}_v))).$$

The famous Shafarevich-Tate conjecture asserts that $\text{III}_{E/K}$ is finite. The main result of these notes is that its finiteness does imply the parity conjecture. (Over function fields, Artin and Tate [58] showed that the full Birch-Swinnerton-Dyer conjecture follows from finiteness of III .)

We write $E[n]$ for the n -torsion in $E(\bar{K})$, and $K(E[n])$ for the field obtained by adjoining to K the coordinates of points in $E[n]$.

Theorem A. *Let E be an elliptic curve over a number field K , and suppose that $\text{III}_{E/K(E[2])}$ has finite 2- and 3-primary parts. Then*

$$(-1)^{\text{rk } E/K} = w(E/K).$$

The problem that finiteness of III implies parity has a reasonably long history. It was solved for elliptic curves over \mathbb{Q} with j -invariant 0 or 1728 by Birch and Stephens [3], for CM elliptic curves over \mathbb{Q} by Greenberg [24] and Guo [25], all (modular) elliptic curves over \mathbb{Q} by Monsky [39] and most modular elliptic curves over totally real fields by Nekovář [41]. Over arbitrary number fields the theorem above is proved in [12] for elliptic curves with ‘decent’ reduction types at 2 and 3, and for all elliptic curves in [16].

1.2. Birch-Swinnerton-Dyer II and isogeny invariance. To explain our approach to the parity conjecture, we need to state the second part of the Birch-Swinnerton-Dyer conjecture.

Notation 1.1. Denote the discriminant of K by Δ_K . For an elliptic curve E/K write $E(K)_{\text{tors}}$ for the *torsion* subgroup of $E(K)$ and

$$\text{Reg}_{E/K} = \det(\langle P_i, P_j \rangle)$$

for the *regulator* of E/K ; here $\{P_i\}$ is any basis of $E(K)/E(K)_{\text{tors}}$ and \langle, \rangle the Néron-Tate height pairing over K .

Finally, we define the product $C_{E/K}$ of *Tamagawa numbers* and *periods*. Fix an invariant differential ω on E , and let ω_v^o denote the Néron differential at a finite place v of K . Set

$$C_{E/K} = \prod_{v \nmid \infty} c_v \left| \frac{\omega}{\omega_v^o} \right|_v \cdot \prod_{\substack{v \mid \infty \\ \text{real}}} \int_{E(K_v)} |\omega| \cdot \prod_{\substack{v \mid \infty \\ \text{cplx}}} 2 \int_{E(K_v)} \omega \wedge \bar{\omega},$$

with c_v the local Tamagawa number at v and $|\cdot|_v$ the normalised absolute value on K_v . We sometimes denote the individual terms of $C_{E/K}$ by

$C(E/K_v, \omega)$, so that $C_{E/K} = \prod_v C(E/K_v, \omega)$. The terms depend on the choice of ω but their product does not: replacing ω by $\alpha\omega$ with $\alpha \in K^\times$ changes it by $\prod_v |\alpha|_v$, which is 1 by the product formula.

Birch–Swinnerton-Dyer Conjecture II. *The Tate-Shafarevich group $\text{III}_{E/K}$ is finite, and the leading coefficient of $L(E/K, s)$ at $s = 1$ is*

$$\frac{|\text{III}_{E/K}| \text{Reg}_{E/K} C_{E/K}}{|E(K)_{\text{tors}}|^2 \sqrt{|\Delta_K|}} =: \text{BSD}_{E/K}.$$

Over general number fields, the only thing known about $\text{BSD}_{E/K}$ is that it is an isogeny invariant, which is a theorem of Cassels [4]. If $\phi : E \rightarrow E'$ is an isogeny over K , then $L(E/K, s) = L(E'/K, s)$ (ϕ induces an isomorphism $V_l(E) \rightarrow V_l(E')$, and the L -function is defined in terms of V_l), so in effect this means that BSD II is compatible with isogenies. Actually, because isogenous curves have the same rank (ϕ induces $E(K) \otimes \mathbb{Q} \cong E'(K) \otimes \mathbb{Q}$), so is BSD I as well. We will need a slight generalisation of Cassels' theorem, which relies on isogeny invariance of BSD II for general abelian varieties:

Lemma 1.2. *Suppose K_i are number fields, E_i/K_i are elliptic curves and n_i are integers. If $\prod_i L(E_i/K_i, s)^{n_i} = 1$ and all III_{E_i/K_i} are finite, then $\prod_i \text{BSD}_{E_i/K_i}^{n_i} = 1$.*

Proof. Taking the terms with $n_i < 0$ to the other side (and renaming the curves and fields if necessary), rewrite the assumed identity of L -functions

$$\prod_i L(E_i/K_i, s) = \prod_j L(E'_j/K'_j, s).$$

Let $A = \prod_i \text{Res}_{K_i/\mathbb{Q}} E_i$ and $A' = \prod_j \text{Res}_{K'_j/\mathbb{Q}} E'_j$ be the products of Weil restrictions of the curves to \mathbb{Q} . These are abelian varieties with $L(A/\mathbb{Q}, s) = L(A'/\mathbb{Q}, s)$, so A and A' have isomorphic l -adic representations (Serre [52] §2.5 Rmk. (3)), and are therefore isogenous (Faltings [19]). Their III 's are assumed to be finite and BSD-quotients are invariant under Weil restriction (Milne [37] §1) and isogeny (Tate–Milne [38] Thm. 7.3). This proves the claim. \square

1.3. Parity example. Why is this relevant to the parity conjecture, which concerns only the first part of BSD? Assume for the moment finiteness of III , and consider the following example. The two elliptic curves over \mathbb{Q}

$$E : y^2 + y = x^3 + x^2 - 7x + 5 \quad \Delta_E = -7 \cdot 13 \quad (91b1)$$

$$E' : y^2 + y = x^3 + x^2 + 13x + 42 \quad \Delta_{E'} = -7^3 \cdot 13^3 \quad (91b2)$$

are isogenous via a 3-isogeny $\phi : E \rightarrow E'$ defined over \mathbb{Q} . Choose ω, ω' ($= \frac{dx}{2y+1}$) to be the global minimal differentials, so $C_p = c_p$ for all p . The curves E, E' have split multiplicative reduction at 7 at 13, and their local Tamagawa numbers and the infinite periods $c_\infty = C(E/\mathbb{R}, \omega)$, $c'_\infty = C(E'/\mathbb{R}, \omega')$ are

$$c_7 = v_7(\Delta_E) = 1, \quad c_{13} = v_{13}(\Delta_E) = 1; \quad c_\infty = 6.039... \\ c'_7 = v_7(\Delta_{E'}) = 3, \quad c'_{13} = v_{13}(\Delta_{E'}) = 1; \quad c'_\infty = 2.013...$$

In fact, $c_\infty = 3c'_\infty$ (see the computation below for general isogenous curves). Now $\text{BSD}_{E/\mathbb{Q}} = \text{BSD}_{E'/\mathbb{Q}}$ by Cassels' theorem, but

$$\left. \begin{aligned} C_{E/\mathbb{Q}} &= 1 \cdot 3 \cdot c_\infty = c_\infty \\ C_{E'/\mathbb{Q}} &= 3 \cdot 3 \cdot \frac{1}{3}c_\infty = 3c_\infty \end{aligned} \right] \text{ not equal,}$$

so some other terms in the Birch–Swinnerton-Dyer constant for E and E' must be unequal as well. Because the two are off by a rational non-square factor, and the conjectural orders of III (as well as of $(\text{torsion})^2$) are squares, the regulators must be unequal! In other words,

$$\frac{\text{Reg}_{E/\mathbb{Q}}}{\text{Reg}_{E'/\mathbb{Q}}} = \frac{C_{E'/\mathbb{Q}}}{C_{E/\mathbb{Q}}} \cdot \frac{|\text{III}_{E'/\mathbb{Q}}|}{|\text{III}_{E/\mathbb{Q}}|} \cdot \frac{|E(\mathbb{Q})_{\text{tors}}|}{|E'(\mathbb{Q})_{\text{tors}}|} = 3 \cdot \square \cdot \square \neq 1.$$

If now $\text{rk } E/\mathbb{Q}$ were 0, then so would be $\text{rk } E'/\mathbb{Q}$ and we would have

$$\text{Reg}_{E/\mathbb{Q}} = \text{Reg}_{E'/\mathbb{Q}} = 1,$$

contradicting the above. So, assuming finiteness of III, we proved that E/\mathbb{Q} has positive rank. In fact, E/\mathbb{Q} has odd rank:

Lemma 1.3. *Let $\phi : E/K \rightarrow E'/K$ be a K -rational isogeny of degree d . Then*

$$\frac{\text{Reg}_{E/K}}{\text{Reg}_{E'/K}} = d^{\text{rk } E/K} \cdot (\text{rational square}).$$

Proof. Write $n = \text{rk } E/K = \text{rk } E'/K$, and pick a basis

$$\Lambda := E(K)/\text{torsion} = \langle P_1, \dots, P_n \rangle, \quad \Lambda' := E'(K)/\text{torsion}.$$

Write $\phi^t : E' \rightarrow E$ for the dual isogeny, so that $\phi^t \phi = [d]$. Then

$$\begin{aligned} d^n \text{Reg}_{E/K} &= \det \langle dP_i, P_j \rangle = \det \langle \phi^t \phi P_i, P_j \rangle \\ &= \det \langle \phi P_i, \phi P_j \rangle = \text{Reg}_{E'/K} \cdot [\Lambda' : \phi(\Lambda)]^2. \end{aligned}$$

□

Returning to our example, we proved that

$$3 \cdot \square = \frac{\text{Reg}_{E/\mathbb{Q}}}{\text{Reg}_{E'/\mathbb{Q}}} = 3^{\text{rk } E/\mathbb{Q}} \cdot \square,$$

so the rank is odd. Because $w(E/\mathbb{Q}) = (-1)^3 = -1$ (2 split places + 1 infinite; see §3), we showed that

$$\text{Finiteness of III} \implies \text{Parity Conjecture}$$

for the curve $E = 91b1/\mathbb{Q}$.

This is the main idea behind the proof of Theorem A in general. We will use equalities of L -functions to deduce relations between rank parities and the C 's, and then verify by means of local computations that the latter agree with the required root numbers.

2. THE p -PARITY CONJECTURE

To prove Theorem A, along the way we will also need slightly finer statements that give unconditional results for the Selmer ranks of E/K .

Recall that III is an abelian torsion group, and for every prime p its p -primary part can be written as

$$\text{III}_{E/K}[p^\infty] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\delta_p} \times (\text{finite } p\text{-group of square order}).$$

Definition. With δ_p as above, the p -infinity Selmer rank of E/K is

$$\text{rk}_p E/K = \text{rk } E/K + \delta_p.$$

If III is finite as expected, then $\delta_p = 0$ and $\text{rk}_p = \text{rk}$ for all p . But the point is that we can often say something about rk_p without assuming finiteness of III . In particular, the following version of the parity conjecture is often accessible:

p -Parity Conjecture. $(-1)^{\text{rk}_p E/K} = w(E/K)$.

This conjecture is known over \mathbb{Q} ([39, 41, 28, 12]) and for ‘most’ elliptic curves over totally real fields [42, 16, 43]; see Theorem 4.4 below.

Remark 2.1. It is more conventional to define rk_p as follows. Let

$$X_p(E/K) = \text{Hom}_{\mathbb{Z}_p}(\varinjlim_n \text{Sel}_{p^n}(E/K), \mathbb{Q}_p/\mathbb{Z}_p)$$

be the Pontryagin dual of the p^∞ -Selmer group of E/K . This is a finitely generated \mathbb{Z}_p -module, and

$$\mathcal{X}_p(E/K) = X_p(E/K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a \mathbb{Q}_p -vector space, whose dimension is precisely rk_p . Both X_p and \mathcal{X}_p are (contravariantly) functorial in E and they behave well under field extensions. Specifically, if F/K is a finite Galois extension, its Galois group acts on $\mathcal{X}_p(E/F)$, and there is a canonical isomorphism of the Galois invariants with $\mathcal{X}_p(E/K)$,

$$(2.2) \quad \mathcal{X}_p(E/K) = \mathcal{X}_p(E/F)^{\text{Gal}(F/K)}.$$

(The proof is a simple inflation-restriction argument, see e.g. [12] Lemma 4.14.). For $p \nmid [F : K]$ this is even true on the level of X_p . Because the same Galois invariance holds for $E(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p \subset \mathcal{X}_p(E/K)$, we get the following corollary (which can also be proved directly).

Corollary 2.3. *Suppose F/K is a finite extension of number fields, and E/K is an elliptic curve. If $\text{III}_{E/F}$ is finite, then so is $\text{III}_{E/K}$.*

Remark 2.4. The definitions above and Corollary 2.3 apply also to abelian varieties in place of elliptic curves.

2.1. Proof of Theorem A. To establish Theorem A, we will need to prove the 2-parity conjecture for elliptic curves with a K -rational 2-torsion point and a special case of the 3-parity conjecture:

Theorem 2.5 (=Theorem 5.1). *Let K be a number field, and E/K an elliptic curve with a K -rational 2-torsion point $O \neq P \in E(K)[2]$. Then*

$$(-1)^{\text{rk}_2 E/K} = (-1)^{\text{ord}_2 \frac{C_{E/K}}{C_{E'/K}}} = w(E/K),$$

where $E' = E/\{O, P\}$ is the 2-isogenous curve.

Theorem 2.6. *Let F/K be an S_3 -extension of number fields, and let M and L be intermediate fields of degree 2 and 3 over K , respectively. For every elliptic curve E over K ,*

$$(-1)^{\text{rk}_3 E/K + \text{rk}_3 E/M + \text{rk}_3 E/L} = (-1)^{\text{ord}_3 \frac{C_{E/F} C_{E/K}^2}{C_{E/M} C_{E/L}^2}} = w(E/K)w(E/M)w(E/L).$$

These two parity statements are sufficient to deduce Theorem A:

Proof of Theorem A. Take $F = K(E[2])$, so $\text{Gal}(F/K) \subset \text{GL}_2(\mathbb{F}_2) \cong S_3$. By Corollary 2.3, the 2- and 3-primary parts of $\text{III}_{E/k}$ are finite for $K \subset k \subset F$.

If E has a K -rational 2-torsion point, the result follows from Theorem 2.5. If F/K is cubic, then $\text{rk } E/K$ and $\text{rk } E/F$ have the same parity, because $E(F) \otimes \mathbb{Q}$ is a rational $\text{Gal}(F/K) \cong C_3$ -representation, so its dimension has the same parity as that of its C_3 -invariants. Also, $w(E/K) = w(E/F)$ (see [31] p.167, or prove this directly using the results of §3), so the result again follows.

We are left with the case when $\text{Gal}(F/K) \cong S_3$. Let M be the quadratic extension of K in F and L one of the cubic ones. By the above argument,

$$\begin{aligned} \text{rk } E/M \text{ even} &\iff w(E/M) = 1, \\ \text{rk } E/L \text{ even} &\iff w(E/L) = 1. \end{aligned}$$

On the other hand, by Theorem 2.6,

$$\text{rk } E/K + \text{rk } E/M + \text{rk } E/L \text{ is even} \iff w(E/K)w(E/M)w(E/L) = 1,$$

and Theorem A is proved. \square

2.2. Local formulae for the Selmer parity. There are two totally different steps in proving Theorems 2.5 and 2.6, one global and one local. The first one is to relate the Selmer parity to the products of periods and Tamagawa numbers (the first equality in the two theorems), and the second one is a local computation comparing the latter to root numbers (the second equality).

The first one is in the spirit of the example in §1.3. Recall that there we had two elliptic curves over \mathbb{Q} and an equality of L -functions,

$$L(E/\mathbb{Q}, s) = L(E'/\mathbb{Q}, s),$$

that originated from an isogeny $E \rightarrow E'$, say of degree d . Assuming finiteness of III, it implies an equality of the Birch–Swinnerton-Dyer quotients $\text{BSD}_{E/\mathbb{Q}} = \text{BSD}_{E'/\mathbb{Q}}$, which reads modulo squares

$$\frac{\text{Reg}_{E'/\mathbb{Q}}}{\text{Reg}_{E/\mathbb{Q}}} \equiv \frac{C_{E/\mathbb{Q}}}{C_{E'/\mathbb{Q}}} \pmod{\mathbb{Q}^{\times 2}}.$$

The ‘lattice index’ argument (Lemma 1.3) shows that the left-hand side is the same as $d^{\text{rk } E/\mathbb{Q}}$ modulo squares. So we have an expression for the parity of $\text{rk } E/\mathbb{Q}$, which is a difficult *global* invariant in terms of easy *local* data. This is an absolutely crucial step in all parity-related proofs. It can be made slightly finer, without assuming finiteness of III, but at the expense of working with Selmer groups. The precise statement is as follows:

Notation 2.7. For an isogeny $\phi : A \rightarrow A'$ of abelian varieties over K we write $\phi^t : (A')^t \rightarrow A^t$ for the dual isogeny. For a prime p write

$$\phi_p : \mathcal{X}_p(A'/K) \rightarrow \mathcal{X}_p(A/K)$$

for the induced map on the dual Selmer groups, and

$$\phi_{p,v} : A(K_v) \rightarrow A'(K_v)$$

for the map on local points. We let $\chi(\cdot) := |\ker(\cdot)|/|\text{coker}(\cdot)|$.

Theorem 2.8. *Let $\phi : A \rightarrow A'$ be a non-zero isogeny of abelian varieties defined over a number field K . For every prime p ,*

$$\frac{\chi(\phi_p^t)}{\chi(\phi_p)} = p\text{-part of } \prod_v \chi(\phi_{p,v}) = p\text{-part of } \frac{C_{A/K}}{C_{A'/K}}.$$

The proof is an application of Poutou-Tate duality (see e.g. [12] §4.1). For example, if $\phi : E \rightarrow E'$ is an isogeny of elliptic curves of prime degree p , then

$$\frac{C_{E/K}}{C_{E'/K}} = \frac{\chi(\phi_p^t)}{\chi(\phi_p)} \equiv \chi(\phi_p^t)\chi_p(\phi_p) = \chi([p]_p) = p^{\text{rk}_p E/K} \pmod{\mathbb{Q}^{\times 2}},$$

which is a formula of Cassels (see Birch [2] or Fisher [20]). This extends the argument of §1.3 and in particular Lemma 1.3 to an unconditional statement for Selmer groups, and proves the first equality of Theorem 2.5 (with $p = 2$).

2.3. Parity in S_3 -extensions. We now proved the first ‘global’ step of theorem 2.5. Now we do the same for Theorem 2.6, in other words we claim the following¹

¹ This is [12] Thm. 4.11 with $p = 3$. Note that the contributions from $v|\infty$ to $\frac{C_{E/F}C_{E/K}^2}{C_{E/M}C_{E/L}^2}$ cancel when using the same K -rational ω over each field. The definition of C in [12] excludes infinite places, so the formula there does not need the $\frac{C_{E/K}^2}{C_{E/L}^2}$ term, as it is then a rational square.

Theorem 2.9. *Let F/K be an S_3 -extension of number fields, M and L intermediate fields of degree 2 and 3 over K , and E/K an elliptic curve. Then*

$$\mathrm{rk}_3 E/K + \mathrm{rk}_3 E/M + \mathrm{rk}_3 E/L \equiv \mathrm{ord}_3 \frac{C_{E/F} C_{E/K}^2}{C_{E/M} C_{E/L}^2} \pmod{2}.$$

As it was done in §1.3 for the isogeny case, we start with a slightly simpler version first, assuming finiteness of III. Again, we will use a relation between L -functions, but this time it is one of a different nature.

Thus, suppose $G = \mathrm{Gal}(F/K) \cong S_3$ and M and L are as in the theorem. Let E/K be an elliptic curve for which III $_{E/F}$ is finite. By Corollary 2.3, III $_{E/K}$, III $_{E/M}$ and III $_{E/L}$ are finite as well.

The group S_3 has 3 irreducible representations, namely $\mathbf{1}$ (trivial), ϵ (sign) and a 2-dimensional representation ρ , all defined over \mathbb{Q} . The list of subgroups of S_3 up to conjugacy is

$$\mathcal{H} = \{1, C_2, C_3, S_3\},$$

and they correspond by Galois theory to F, L, M and K respectively. Each $H \in \mathcal{H}$ gives rise to a representation $\mathbb{C}[G/H]$ of G , associated to the G -action on the left cosets of H in G . Because there are *four* subgroups and only *three* irreducible representations, there is a relation between these. Writing out

$$\mathbb{C}[G] \cong \mathbf{1} \oplus \epsilon \oplus \rho \oplus \rho, \quad \mathbb{C}[G/C_3] \cong \mathbf{1} \oplus \epsilon \quad \text{and} \quad \mathbb{C}[G/C_2] \cong \mathbf{1} \oplus \rho,$$

we find that the (unique such) relation is

$$(2.10) \quad \mathbb{C}[S_3] \oplus \mathbb{C}[S_3/S_3]^{\oplus 2} \cong \mathbb{C}[S_3/C_3] \oplus \mathbb{C}[S_3/C_2]^{\oplus 2}.$$

Now tensor this relation with the l -adic representation $V_l(E/K)_{\mathbb{C}} = V_l(E/K) \otimes_{\mathbb{Q}_l} \mathbb{C}$ (embedding $\mathbb{Q}_l \hookrightarrow \mathbb{C}$ in some way). By the Artin formalism for L -functions (see §3), for every $H \in \mathcal{H}$,

$$L(V_l(E/K)_{\mathbb{C}} \otimes \mathbb{C}[G/H], s) = L(E/F^H, s),$$

so we have a relation between L -functions²,

$$L(E/F, s) L(E/K, s)^2 = L(E/M, s) L(E/L, s)^2.$$

Applying Lemma 1.2, we get a relation between the regulators and the products of periods and Tamagawa numbers,

$$(2.11) \quad \frac{\mathrm{Reg}_{E/F} \mathrm{Reg}_{E/K}^2}{\mathrm{Reg}_{E/M} \mathrm{Reg}_{E/L}^2} \equiv \frac{C_{E/M} C_{E/L}^2}{C_{E/F} C_{E/K}^2} \pmod{\mathbb{Q}^{\times 2}}.$$

How do we interpret the left-hand side, and why is it even a rational number? Tensor the Mordell-Weil group $E(F)$ with \mathbb{Q} and decompose it as a G -representation,

$$V = E(F) \otimes \mathbb{Q} \cong \mathbf{1}^{\oplus a} \oplus \epsilon^{\oplus b} \oplus \rho^{\oplus c}.$$

²The elliptic curve has nothing to do with this: this relation already exists on the level of Dedekind zeta-functions of the number fields. This forces relations between the regulators and class groups of number fields, and they have been studied by Brauer, Kuroda, de Smit and others (see e.g. [1] for references).

Next, compute the ranks of E over the intermediate fields of F/K in terms of a , b and c : using Frobenius reciprocity, for a subgroup H of S_3 we have

$$\mathrm{rk} E/F^H = \dim V^H = \langle \mathbf{1}, V|_H \rangle_H = \langle \mathbb{C}[G/H], V \rangle_G = \begin{cases} a, & H = S_3. \\ a+b, & H = C_3 \\ a+c, & H = C_2 \\ a+b+2c, & H = \{1\}. \end{cases}$$

So, let P_1, \dots, P_a be a basis of $E(K) \otimes \mathbb{Q}$ (the ‘trivial’ part), and complement it to a basis of $E(M) \otimes \mathbb{Q}$ with Q_1, \dots, Q_b (the ‘sign’ part) and to a basis of $E(L) \otimes \mathbb{Q}$ with R_1, \dots, R_c . By clearing the denominators, we may assume that the P ’s, Q ’s and R ’s are actual points in $E(F)$. If $g \in \mathrm{Gal}(F/K)$ is an element of order 3, then

$$E(F) \otimes \mathbb{Q} = \langle P_1, \dots, P_a, Q_1, \dots, Q_b, R_1, \dots, R_c, R_1^g, \dots, R_c^g \rangle,$$

in other words these points form a basis of a subgroup of $E(F)$ of finite index. Now we can compute all the regulators. Consider the three determinants

$$\mathcal{P} = \det(\langle P_i, P_j \rangle_F), \quad \mathcal{Q} = \det(\langle Q_i, Q_j \rangle_F), \quad \text{and} \quad \mathcal{R} = \det(\langle R_i, R_j \rangle_F),$$

where $\langle \cdot, \cdot \rangle_F$ is the Néron-Tate height pairing for E/F . Recall that the pairing is normalised in such a way that it changes if computed over a different field by the degree of the field extension. For instance,

$$\langle P_i, P_j \rangle_K = \frac{1}{6} \langle P_i, P_j \rangle_F,$$

so $\mathrm{Reg}_{E/K}$ is $(\frac{1}{6})^a \cdot \mathcal{P} \cdot \square$, where \square is the inverse square of the index of the lattice spanned by the P_i in $E(K)$. As we are only interested in regulators *modulo squares*, we may ignore this. Similarly,

$$\mathrm{Reg}_{E/M} = \left(\frac{1}{3}\right)^{a+b} \mathcal{P} \mathcal{Q} \cdot \square \quad \text{and} \quad \mathrm{Reg}_{E/L} = \left(\frac{1}{2}\right)^{a+c} \mathcal{P} \mathcal{R} \cdot \square,$$

because all $\langle P_i, Q_j \rangle_F$ and $\langle P_i, R_j \rangle_F$ are 0, so both regulators are really products of two determinants. (The height pairing is Galois-invariant, so different isotypical components are always orthogonal to each other with respect to it.) Finally, using Galois invariance again, together with the fact that $R_i + R_i^g + R_i^{g^2}$ is S_3 -invariant and so orthogonal to R_j , we find that $\langle R_i, R_j^g \rangle_F = -\frac{1}{2} \langle R_i, R_j \rangle_F$, so

$$\mathrm{Reg}_{E/F} = \mathcal{P} \cdot \mathcal{Q} \cdot \det \begin{pmatrix} A & -\frac{1}{2}A \\ -\frac{1}{2}A & A \end{pmatrix} \cdot \square = \mathcal{P} \cdot \mathcal{Q} \cdot 3^c \mathcal{R}^2 \cdot \square,$$

where A is the matrix $(\langle R_i, R_j \rangle)_{i,j}$. Combining the four regulators yields

$$\begin{aligned} \frac{\mathrm{Reg}_{E/F} \mathrm{Reg}_{E/K}^2}{\mathrm{Reg}_{E/M} \mathrm{Reg}_{E/L}^2} &\equiv \frac{3^c \mathcal{P} \mathcal{Q} \mathcal{R}^2 \cdot (6^a \mathcal{P})^2}{3^{a+b} \mathcal{P} \mathcal{Q} \cdot (2^{a+c} \mathcal{P} \mathcal{R})^2} \equiv 3^{a+b+c} \\ &\equiv 3^{a+(a+b)+(a+c)} \\ &\equiv 3^{\mathrm{rk} E/K + \mathrm{rk} E/M + \mathrm{rk} E/L} \pmod{\mathbb{Q}^{\times 2}}. \end{aligned}$$

Together with (2.11) this proves Theorem 2.9, assuming finiteness of III.

2.4. Brauer relations and regulator constants. The results of §2.3 generalise to arbitrary Galois groups as follows (see [12, 14] for details).

Definition 2.12. Let G be a finite group, and write \mathcal{H} for the set of representatives of subgroups of G up to conjugacy. A formal linear combination

$$\Theta = \sum_i n_i H_i \quad (n_i \in \mathbb{Z}, H_i \in \mathcal{H})$$

is a *Brauer relation* if $\oplus_i \mathbb{C}[G/H_i]^{\oplus n_i} = 0$ as a virtual representation, i.e. if the character $\sum_i n_i \operatorname{Ind}_{H_i}^G \mathbf{1}_{H_i}$ is zero.

Example 2.13. The dihedral group $G = D_{2p}$ for an odd prime p has a relation

$$\Theta = \{1\} - 2C_2 - C_p + 2G,$$

the only one in G up to multiples. For $p = 3$ this is the relation (2.10).

If $G = \operatorname{Gal}(F/K)$ is a Galois group and E/K is an elliptic curve, every Brauer relation $\Theta = \sum n_i H_i$ in G gives an identity of L -functions

$$\prod_i L(E/M_i, s)^{n_i} = 1 \quad (M_i = F^{H_i}),$$

and so a relation like (2.11) between regulators and Tamagawa numbers, assuming that $\operatorname{III}_{E/F}$ is finite. As in the case $G = S_3$, the left-hand side $\prod_i \operatorname{Reg}_{E/M_i}^{n_i}$ depends only on $E(F) \otimes \mathbb{Q}$ as a G -representation:

Theorem 2.14. *Let G be a finite group and $\Theta = \sum n_i H_i$ a Brauer relation in G . Suppose V is a $\mathbb{Q}G$ -representation and $\langle, \rangle : V \times V \rightarrow \mathbb{R}$ is a non-degenerate bilinear G -invariant pairing³. Then*

$$\mathcal{C}_\Theta(V) := \prod_i \det\left(\frac{1}{|H_i|} \langle, \rangle \mid V^{H_i}\right)^{n_i}$$

is a well-defined number in $\mathbb{Q}^\times / \mathbb{Q}^{\times 2}$, and it is independent of the choice of the pairing \langle, \rangle .

The notation $\det\left(\frac{1}{|H_i|} \langle, \rangle \mid V^{H_i}\right)$ means the following: pick a basis $\{P_j\}$ of the invariant subspace V^{H_i} and compute the determinant of the matrix whose entries are $\frac{1}{|H_i|} \langle P_j, P_k \rangle$. Up to rational squares, this is independent on the basis choice and the total product is well-defined in $\mathbb{R}^\times / \mathbb{Q}^{\times 2}$. The theorem asserts that it is in fact in $\mathbb{Q}^\times / \mathbb{Q}^{\times 2}$, and independent of \langle, \rangle .

Remark 2.15. The theorem also holds with \mathbb{Q} replaced by any other field k where $|G|$ is invertible, V by a self-dual kG -representation and \mathbb{R} by any field containing k .

The proof of the theorem is reasonably straightforward (see [12] §2 or [14] §2.ii), in the spirit of what we did in §2.3 explicitly for $G = S_3$. An immediate consequence is that if ρ_1, \dots, ρ_k are all irreducible $\mathbb{Q}G$ -representations,

³It exists for every such V , as every rational representation is self-dual

then the numbers $\mathcal{C}_\Theta(\rho_1), \dots, \mathcal{C}_\Theta(\rho_k)$, called *regulator constants*, determine everything. In other words, if we decompose $V = \bigoplus \rho_j^{\oplus a_j}$, then

$$\mathcal{C}_\Theta(V) = \prod_j \mathcal{C}_\Theta(\rho_j)^{a_j},$$

as we can obviously pick a ‘diagonal’ pairing on V which respects the decomposition.

Example 2.16. For $G = S_3$ and $\Theta = \{1\} - 2C_2 - C_3 + 2S_3$, we have

$$\mathcal{C}_\Theta(\mathbf{1}) = \mathcal{C}_\Theta(\epsilon) = \mathcal{C}_\Theta(\rho) = 3 \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2}.$$

(For example, if we choose the obvious trivial pairing on $\mathbf{1}$, then

$$\mathcal{C}_\Theta(\mathbf{1}) = 1 \cdot \left(\frac{1}{2}\right)^{-2} \cdot \left(\frac{1}{3}\right)^{-1} \cdot \left(\frac{1}{6}\right)^2 \equiv 3 \pmod{\mathbb{Q}^\times / \mathbb{Q}^{\times 2}},$$

and similarly for ϵ and ρ .)

Corollary 2.17. *Suppose E/K is an elliptic curve, F/K a Galois extension with Galois group G and $\Theta = \sum n_i H_i$ a Brauer relation in G . Decompose $E(F) \otimes \mathbb{Q} = \bigoplus \rho_k^{\oplus a_k}$ into irreducible G -representations. Then*

$$\prod_i \text{Reg}_{E/F^{H_i}}^{n_i} = \prod_k \mathcal{C}_\Theta(\rho_k)^{a_k},$$

Proof. Take $V = E(F) \otimes \mathbb{Q}$ and \langle, \rangle the Néron-Tate height pairing on V . \square

For $G = S_3$, together with (2.11) this corollary reproves Theorem 2.9, still assuming that $\text{III}_{E/F}$ is finite.

Finally, we discuss the modification necessary to turn this into an unconditional statement about Selmer ranks. Suppose E/K is an elliptic curve, $\text{Gal}(F/K) = G$ as before and p is a prime. The dual Selmer $\mathcal{X}_p(E/F)$ (cf. Remark 2.1) is a $\mathbb{Q}_p G$ -representation which now plays a role analogous to the $\mathbb{Q}G$ -representation $E(F) \otimes \mathbb{Q}$.

Theorem 2.18.

- (1) $\mathcal{X} = \mathcal{X}_p(E/F)$ is a self-dual $\mathbb{Q}_p G$ -representation. In other words, it possesses a G -invariant \mathbb{Q}_p -valued non-degenerate bilinear pairing.
- (2) For any such pairing \langle, \rangle and a Brauer relation $\Theta = \sum n_i H_i$ in G , we have

$$\text{ord}_p \left(\prod_i \det \left(\frac{1}{|H_i|} \langle, \rangle \mid \mathcal{X}^{H_i} \right)^{n_i} \right) \equiv \text{ord}_p \prod_i C_{E/F^{H_i}}^{n_i} \pmod{2}.$$

For the proof see [13]. In fact, for an explicit group such as $G = S_3$ it is possible to give a direct proof of this, by constructing an isogeny between the products of the relevant Weil restrictions and applying Theorem 2.8; this is how it is done in [12] Thm. 4.11 (with $p = 3$). Note that when $G = S_3$, every G -representation is rational, so (1) is automatic.

As a corollary, we get Theorem 2.9. (Decompose $\mathcal{X}_3(E/F)$ into $\mathbb{Q}_p S_3$ -irreducibles and apply the theorem above.) So we have proved the first equalities in Theorems 2.5 and 2.6, and it remains to prove the second ones.

For that we need to look carefully at the root numbers of elliptic curves and carry out the local computations relating them to Tamagawa numbers.

2.5. Parity in dihedral extensions. The S_3 -example generalises to dihedral groups. Suppose $G = \text{Gal}(F/K) = D_{2p}$, with p an odd prime, and recall from Example 2.13 that G has a unique Brauer relation

$$\Theta = \{1\} - 2C_2 - C_p + 2G.$$

Let M and L be the unique quadratic and one of the degree p extensions of K in F , respectively. Apply Theorem 2.18 and interpret the left-hand side using regulator constants (using Theorem 2.14 and Remark 2.15 with $k = \mathbb{Q}_p$). Take any elliptic curve E/K , and decompose

$$\mathcal{X}_p(E/F) = \mathbf{1}^{\oplus n_1} \oplus \epsilon^{\oplus n_\epsilon} \oplus \rho^{\oplus n_\rho},$$

where $\mathbf{1}$ (trivial), ϵ (sign) and ρ ($p-1$ -dimensional) are the distinct $\mathbb{Q}_p G$ -irreducible representations; the latter decomposes over \mathbb{Q}_p as a sum of 2-dimensionals,

$$\rho \otimes \bar{\mathbb{Q}}_p \cong \tau_1 \oplus \cdots \oplus \tau_{\frac{p-1}{2}}.$$

The regulator constants of $\mathbf{1}$, ϵ and ρ are easily seen to be p (as in the case $G = S_3$), and Theorem 2.18 gives the following parity statement:

$$n_1 + n_\epsilon + n_\rho \equiv \text{ord}_p \frac{C_{E/F} C_{E/K}^2}{C_{E/M} C_{E/L}^2} \pmod{2}.$$

This can also be written as

$$\langle \mathbf{1} + \epsilon + \tau_i, \mathcal{X}_p(E/F) \rangle \equiv \text{ord}_p \frac{C_{E/F} C_{E/K}^2}{C_{E/M} C_{E/L}^2} \pmod{2} \quad \forall i,$$

where \langle, \rangle stands for the usual inner product of characters of representations. In other words, the relation Θ gives a p -parity expression for ‘the twist of E by $\mathbf{1} + \epsilon + \tau_i$ ’ in terms of Tamagawa numbers and periods.

Remark 2.19. For an alternative expression for exactly the same parity in terms of ‘local constants’, see Mazur and Rubin’s papers [34, 35]; the parity conjecture for these twists is now known for all elliptic curves over number fields and all odd p , see [12, 16] and de La Rochefoucauld [47].

2.6. The Kramer-Tunnell theorem. We mentioned two types of relations between L -functions of elliptic curves: one comes from a rational isogeny, and one from Brauer relations in Galois groups. There is a third example, which is classical: the relation for quadratic twists.

Suppose $M = K(\sqrt{\beta})$ is a quadratic extension of number fields, E/K is an elliptic curve, and E_β/K is the quadratic twist of E by β :

$$E : y^2 = x^3 + ax + b, \quad E_\beta : \beta y^2 = x^3 + ax + b \quad (\cong y^2 = x^3 + \beta^2 ax + \beta^3 b).$$

The l -adic Tate modules of E and E_β are related by

$$T_l(E_\beta) = T_l(E) \otimes \epsilon,$$

with $\epsilon : \text{Gal}(M/K) \rightarrow \{\pm 1\}$ the non-trivial character, and Artin formalism of L -functions (see §3) applied to $\text{Ind}_{\text{Gal}(\bar{K}/M)}^{\text{Gal}(\bar{K}/K)} \mathbf{1} = \mathbf{1} \oplus \epsilon$ yields a relation

$$L(E/M, s) = L(E/K, s)L(E_\beta/K, s).$$

If we assume that $\text{III}_{E/M}$ is finite, Lemma 1.2 gives

$$\frac{\text{Reg}_{E/M}}{\text{Reg}_{E/K} \text{Reg}_{E_\beta/K}} \equiv \frac{C_{E/K} C_{E_\beta/K}}{C_{E/M}} \pmod{\mathbb{Q}^{\times 2}},$$

and it is easy to see that the left-hand side is $2^{\text{rk } E/M}$ (i.e. $2^{\text{rk } E/K + \text{rk } E_\beta/K}$) up to squares. In fact, the Weil restriction $A = \text{Res}_{M/K} E$ admits an isogeny

$$\phi : A \rightarrow E \times E_\beta,$$

such that $\phi^t \phi = [2]$, and Theorem 2.8 produces an unconditional version:

$$(2.20) \quad \text{rk}_2 E/M \equiv \text{ord}_2 \frac{C_{E/K} C_{E_\beta/K}}{C_{E/M}} \pmod{2}.$$

This was used by Kramer [30] and Kramer–Tunnell [31] to prove the 2-parity conjecture for E/M , by comparing the right-hand side with the root number $w(E/M)$ by a local computation:

Theorem 2.21 (Kramer–Tunnell). *Suppose the primes of additive reduction for E above 2 are unramified in M/K . Then the 2-parity conjecture holds for E/M :*

$$(-1)^{\text{rk}_2 E/M} = w(E/M).$$

The restrictions on the reduction type can in fact be removed using the methods of §5.7. We will not need this, so we refer the reader to [16] (‘proof of the Kramer–Tunnell conjecture’).

3. L -FUNCTIONS AND ROOT NUMBERS

To set up the notation, let K be a number field, and \mathfrak{p} a prime of K with completion $K_{\mathfrak{p}}$, residue field \mathbb{F}_q of characteristic p and uniformiser π . We write G_K for the absolute Galois group $\text{Gal}(\bar{K}/K)$ of K , and use a similar notation for other fields as well. For the most of this section we work in the local setting, and we begin by recalling the structure of the local Galois group at \mathfrak{p} . The reduction map on automorphisms puts $G_{K_{\mathfrak{p}}}$ into an exact sequence

$$1 \longrightarrow I_{\mathfrak{p}} \longrightarrow G_{K_{\mathfrak{p}}} \xrightarrow{\text{mod } \mathfrak{p}} G_{\mathbb{F}_q} \longrightarrow 1,$$

which defines the *inertia group* $I_{\mathfrak{p}}$ at \mathfrak{p} . An (*arithmetic*) *Frobenius* is any element $\text{Frob}_{\mathfrak{p}} \in G_{K_{\mathfrak{p}}}$ whose reduction mod \mathfrak{p} is the map $x \mapsto x^q$. If we choose an embedding $\bar{K} \hookrightarrow \bar{K}_{\mathfrak{p}}$, then $G_{K_{\mathfrak{p}}} \hookrightarrow G_K$ via the restriction map, and we can consider $I_{\mathfrak{p}}$ as a subgroup of G_K . Choosing a different embedding conjugates $I_{\mathfrak{p}}$, so it is really only well-defined up to conjugation. Similarly, we can view $\text{Frob}_{\mathfrak{p}}$ as an element of G_K , but it is only defined up to conjugation, and only modulo inertia.

We will often use two standard characters that come from identifications $G_{K_{\mathfrak{p}}}/I_{\mathfrak{p}} \cong \prod \mathbb{Z}_l^{\times}$ and $I_{\mathfrak{p}}/\text{inertia}^{\text{wild}} \cong \prod_{l \neq p} \mathbb{Z}_l$. Fix a prime $l \nmid p$.

Definition 3.1. The (l -adic) *cyclotomic character* $\chi : G_{K_{\mathfrak{p}}} \rightarrow \mathbb{Z}_l^{\times}$ is defined by $\chi(I_{\mathfrak{p}}) = 1$ and $\chi(\text{Frob}_{\mathfrak{p}}) = q$. Alternatively, it is the action of $G_{K_{\mathfrak{p}}}$ on the l -power roots of unity,

$$\chi : G_{K_{\mathfrak{p}}} \longrightarrow \varprojlim_n \text{Aut } \mu_{l^n} \cong \varprojlim_n (\mathbb{Z}/l^n \mathbb{Z})^{\times} \cong \mathbb{Z}_l^{\times}.$$

(In this way it can be defined as a character of G_K .)

Definition 3.2. The (l -adic) *tame character* $\phi : I_{\mathfrak{p}} \rightarrow \mathbb{Z}_l$ is defined by

$$\phi(\sigma) = (\sigma(\pi^{1/l^n})/\pi^{1/l^n}) \in \varprojlim_n \mu_{l^n} \cong \mathbb{Z}_l.$$

Recall that a $G_{K_{\mathfrak{p}}}$ -module M is *unramified* if $I_{\mathfrak{p}}$ acts trivially on M , and, similarly, a G_K -module is *unramified at \mathfrak{p}* if $I_{\mathfrak{p}} < G_K$ acts trivially on it.

Example 3.3. The trivial character $\mathbf{1}$ and the cyclotomic character χ both give \mathbb{Z}_l a structure of an unramified $G_{K_{\mathfrak{p}}}$ -module.

3.1. L -functions. Let E/K be an elliptic curve. For every prime l , the Galois group $G_K = \text{Gal}(\bar{K}/K)$ acts on the sets $E[l^n] = E(\bar{K})[l^n]$ of l^n -torsion points of E for all $n \geq 1$. The fundamental arithmetic invariant of E/K is its *l -adic Tate module*,

$$T_l E = \varprojlim_n E[l^n],$$

the limit taken with respect to the multiplication by l maps $E[l^{n+1}] \rightarrow E[l^n]$. This is a free \mathbb{Z}_l -module of rank 2, and

$$V_l E = T_l E \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

is a 2-dimensional \mathbb{Q}_l -vector space. By the non-degeneracy and the Galois equivariance of the Weil pairing on $E[l^n]$, we have $\det V_l E = \chi$. The representation $V_l E$ (or sometimes its dual $V_l E^*$) is called the *l -adic representation* associated to E/K . For varying l these form a *compatible system of l -adic representations*, meaning that they satisfy two conditions:

- (1) All $V_l E$ are unramified at \mathfrak{p} for almost all primes \mathfrak{p} of K .
- (2) For such \mathfrak{p} , the characteristic polynomial of Frobenius $\text{Frob}_{\mathfrak{p}}$ on $V_l E$ is independent of l for $\mathfrak{p} \nmid l$.

To explain this, take a prime \mathfrak{p} of K and any $l \nmid q$. The criterion of Néron-Ogg-Shafarevich ([54] Ch. VII) asserts that

$$V_l E \text{ is unramified at } \mathfrak{p} \iff E \text{ has good reduction at } \mathfrak{p}.$$

In particular, this happens for almost all primes \mathfrak{p} of K and this is independent of l . This is condition (1).

Now, as $I_{\mathfrak{p}} < G_{K_{\mathfrak{p}}}$ and $V_l E$ is a $G_{K_{\mathfrak{p}}}$ -representation, the quotient $G_{K_{\mathfrak{p}}}/I_{\mathfrak{p}}$ acts on the inertia invariants $(V_l E)^{I_{\mathfrak{p}}}$, so we can talk of the action of Frobenius $\text{Frob}_{\mathfrak{p}}$ on $(V_l E)^{I_{\mathfrak{p}}}$, and this is independent of the choice of $\text{Frob}_{\mathfrak{p}}$. Its

characteristic polynomial does not change under conjugation, therefore it is completely choice-independent. So we may define the local polynomial at \mathfrak{p} ,

$$F_{\mathfrak{p}}(T) = \det(1 - \text{Frob}_{\mathfrak{p}}^{-1} T \mid (V_l E^*)^{I_{\mathfrak{p}}}).$$

(There are two technical points: we take the geometric Frobenius $\text{Frob}_{\mathfrak{p}}^{-1}$, the inverse of the arithmetic one, and we also compute the characteristic polynomial on the dual $V_l E^*$; both are just standard conventions, and are not too important.) As explained in [54] Ch. V,

$$E/K_{\mathfrak{p}} \text{ has good reduction} \implies F_{\mathfrak{p}}(T) = 1 - a_{\mathfrak{p}}T + qT^2,$$

with $a_{\mathfrak{p}} = q + 1 - |E(\mathbb{F}_q)|$. This polynomial has degree 2, since $V_l E = (V_l E)^{I_{\mathfrak{p}}}$ in the good reduction case. Also, $F_{\mathfrak{p}}(T)$ is in $\mathbb{Z}[T]$ rather than just $\mathbb{Z}_l[T]$ and is independent of l . This is condition (2).

Definition 3.4. The L -function of E/K is a function of complex variable s given by the Euler product

$$L(E/K, s) = \prod_{\mathfrak{p}} F_{\mathfrak{p}}(q_{\mathfrak{p}}^{-s})^{-1} = \prod_{\mathfrak{p}} \det(1 - q_{\mathfrak{p}}^{-s} \text{Frob}_{\mathfrak{p}}^{-1} \mid (V_l E^*)^{I_{\mathfrak{p}}})^{-1}.$$

Here the two products run over all primes \mathfrak{p} of K , and $q_{\mathfrak{p}}$ denotes the size of the residue field at \mathfrak{p} . (It follows from the Hasse-Weil bound that $L(E/K, s)$ converges for $\text{Re } s > 3/2$.)

In the same way one associates an L -function to any compatible system $\rho = (\rho_l)_l$ of l -adic representations. There are obvious notions of direct sums and induction for l -adic representations and it is not hard to verify that their L -functions satisfy the following:

- (i) If ρ, ρ' are compatible systems of l -adic representations of G_K , then

$$L(\rho \oplus \rho', s) = L(\rho, s)L(\rho', s).$$

- (ii) For ρ as above and a subfield $F \subset K$,

$$L(\text{Ind}_{G_K}^{G_F} \rho, s) = L(\rho, s).$$

These properties are known as the *Artin formalism* for L -functions.

As for elliptic curves, we expect all L -functions of systems of l -adic representations to have meromorphic (and usually analytic) continuation to the whole of \mathbb{C} , and to satisfy a functional equation relating the value at s to that at $k - s$, where $k \in \mathbb{Z}$ is the *weight* of ρ .

Remark 3.5. Most L -functions that we know of are supposed to arise from l -adic representations. For instance, if V/K is any non-singular projective variety and $0 \leq i \leq 2 \dim V$, the étale cohomology groups $H_{\text{ét}}^i(V, \mathbb{Q}_l)$ form a compatible system. Thus we can define the corresponding L -function $L(H^i(V), s)$, and it has weight $k = i + 1$. In this terminology, $L(E/K, s) = L(H^1(E/K), s)$. (This the reason for taking the dual $V_l E^*$ instead of $V_l E$ in definition of $L(E/K, s)$.)

There is one subtle point though: for varieties other than curves and abelian varieties, it is not known that the polynomials $F_{\mathfrak{p}}(T)$ are independent

of l when \mathfrak{p} is a prime of bad reduction. This is conjectured to be true, and this conjecture is implicitly assumed when one speaks of L -functions of general varieties.

Remark 3.6 (Conductors). Another arithmetic invariant that enters the functional equation of any L -function is its conductor N . For an elliptic curve E/K , the conductor of $L(E/K, s)$ is $N = \Delta_K^2 \text{Norm}_{K/\mathbb{Q}}(\mathcal{N}_{E/K})$. Here Δ_K is the discriminant of K , and $\mathcal{N}_{E/K}$ is the conductor of E/K . It is an ideal in \mathcal{O}_K (see [55]), and we interpret its norm to \mathbb{Q} as a positive integer. It is this N that enters the functional equation in §1.1.

For elliptic curves, let us determine $F_{\mathfrak{p}}(T)$ in all cases, and verify its independence of l explicitly. There are several cases:

$E/K_{\mathfrak{p}}$ has good reduction. As mentioned above, $F_{\mathfrak{p}}(T)$ has degree 2, is independent of l and is of the form

$$F_{\mathfrak{p}}(T) = 1 - a_{\mathfrak{p}}T + q_{\mathfrak{p}}T^2.$$

$E/K_{\mathfrak{p}}$ has split multiplicative reduction. In this case, Tate's uniformization ('theory of the Tate curve') asserts that there is a (p -adic analytic) isomorphism of $G_{K_{\mathfrak{p}}}$ -modules

$$E(\overline{K}_{\mathfrak{p}}) \cong \overline{K}_{\mathfrak{p}}^{\times} / a^{\mathbb{Z}}$$

for some $a \in K_{\mathfrak{p}}^{\times}$ with $v_{\mathfrak{p}}(a) = -v_{\mathfrak{p}}(j(E)) > 0$. Writing $\zeta_{l^n} \in \overline{K}_{\mathfrak{p}}^{\times}$ for a primitive l^n th root of unity, we get that in particular,

$$E[l^n] \cong \langle \zeta_{l^n}, a^{1/l^n} \rangle \subset \overline{K}_{\mathfrak{p}}^{\times}$$

as a Galois module. In this basis, the action of Galois on $E[l^n]$ is of the form

$$G_{K_{\mathfrak{p}}} \ni \sigma \longmapsto \begin{pmatrix} \chi(\sigma) & v_{\mathfrak{p}}(a) \cdot * \\ 0 & 1 \end{pmatrix}$$

with χ the l -adic cyclotomic character and $*$ restricting to the tame character ϕ on $I_{\mathfrak{p}}$. Passing to the inverse limit $T_l E = \varprojlim E[l^n]$ and tensoring with \mathbb{Q}_l we find that the actions on $V_l E$ and $V_l E^*$ are

$$\sigma \mapsto \begin{pmatrix} \chi(\sigma) & * \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma \mapsto \begin{pmatrix} \chi^{-1}(\sigma) & 0 \\ * & 1 \end{pmatrix}$$

respectively. The inertia $I_{\mathfrak{p}}$ acts as $\begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}$ on $V_l E^*$, so it has a 1-dimensional invariant subspace, spanned by the second basis vector. Frobenius acts trivially on it, so

$$F_{\mathfrak{p}}(T) = \det(1 - \text{Frob}_{\mathfrak{p}}^{-1} T \mid (V_l E^*)^{I_{\mathfrak{p}}}) = 1 - T.$$

E/K_p has non-split multiplicative reduction. Let $K_p(\sqrt{\xi})/K_p$ be the unramified quadratic extension, and

$$\eta: G_{K_p} \twoheadrightarrow \text{Gal}(K_p(\sqrt{\xi})/K_p) \xrightarrow{\sim} \{\pm 1\}$$

the associated character of order 2. The quadratic twist E_ξ of E has split-multiplicative reduction and $V_l E = V_l(E_\xi) \otimes \eta$ as a Galois module. In other words, G_{K_p} acts on $V_l E^*$ as

$$\sigma \mapsto \begin{pmatrix} \chi^{-1}(\sigma)\eta(\sigma) & 0 \\ * & \eta(\sigma) \end{pmatrix}$$

Because I_p acts in the same way as in the split multiplicative reduction case and $\eta(\text{Frob}_p) = -1$, we find that the inertia invariants are again 1-dimensional and

$$F_p(T) = 1 + T.$$

E/K_p has additive potentially multiplicative reduction. This is similar to the non-split multiplicative case, except that here $K(\xi)/K$ is replaced by a ramified quadratic extension (namely, the smallest extension where E acquires split multiplicative reduction). Now I_p acts through $\pm \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ on $V_l E^*$ with non-trivial \pm , and has therefore trivial invariants. So

$$F_p(T) = 1.$$

E/K_p has additive potentially good reduction. As in the last case, we claim that $(V_l E^*)^{I_p} = 0$ and consequently

$$F_p(T) = 1.$$

Recall that E acquires good reduction after a finite extension of K_p . By the Néron-Ogg-Shafarevich criterion, I_p acts on $V_l E$ non-trivially and through a finite quotient. If $V_l E$ had a 1-dimensional inertia invariant subspace, then I_p would act as $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ in some basis. The bottom right corner is 1 as $\det V_l E = \chi$ is unramified, and the top-right corner is then 0 since an action of a finite group is diagonalizable. So I_p would act trivially, a contradiction. Therefore $(V_l E^*)^{I_p} = 0$, as asserted.

3.2. Weil-Deligne representations. Whatever the reduction type of E/K_p is, note that I_p has an open (i.e. finite index) subgroup which acts unipotently on $V_l E$, namely trivially in the potentially good case and as $\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}$ in the potentially multiplicative case.

Grothendieck's monodromy theorem asserts that for any non-singular projective variety V and any i , the action of some open subgroup of I_p on $\rho = H_{\text{ét}}^i(V)$ is unipotent. It follows that it has the form $1 + \phi N$ for some fixed nilpotent endomorphism N of ρ . We are going to call such representations Weil representations ($N = 0$) and Weil-Deligne representations (any N). There are two technical points: one is that we fix some embedding $\mathbb{Q}_l \hookrightarrow \mathbb{C}$ and make our representations complex instead of l -adic from this point onwards; another one is that to get rid of the dependence of the l -adic

tame character ϕ of l , we just remember the nilpotent endomorphism N and how it commutes with the Galois group.

Definition 3.7. A *Weil representation* over F of dimension n is a homomorphism $G_F \rightarrow \mathrm{GL}_n(\mathbb{C})$ whose kernel contains a finite index open subgroup of the inertia group $I_{\bar{F}/F}$.

Definition 3.8. A *Weil-Deligne representation* over F is a Weil representation $\rho : G_F \rightarrow \mathrm{GL}_n(V)$ together with a nilpotent endomorphism $N \in \mathrm{End}(V)$ such that $\rho(g)N\rho(g)^{-1} = \chi(g)N$ for all $g \in G_F$.

Example 3.9. If E/K_p is an elliptic curve, then $V_l E \otimes \mathbb{C}$ and $(V_l E^*) \otimes \mathbb{C}$ have a natural structure of Weil-Deligne representations. They are Weil representations if and only if E has integral j -invariant (equivalently, E has potentially good reduction).

3.3. Epsilon-factors. The current state of affairs is that we are very far from proving the Hasse-Weil conjecture for compatible systems of l -adic representations or even for elliptic curves over number fields. Even for elliptic curves over \mathbb{Q} the proof (via modularity) is rather roundabout. In some sense, the only well-understood situation is the 1-dimensional case, that of Hecke characters (a.k.a. ‘Größencharakteren’). Also well-understood are the signs in the conjectural functional equations for all L -functions. This is the theory of ϵ -factors, which we now sketch.

We will not need Hecke characters themselves, only their local components. Let \mathfrak{p} be a prime of K , and denote by F some finite extension of K_p .

Definition 3.10. A *quasi-character* of G_F is a one-dimensional Weil-Deligne (equivalently, Weil) representation. Alternatively, it is a homomorphism

$$\psi : G_F \rightarrow \mathbb{C}^\times,$$

which is continuous with respect to the profinite topology on G_F and discrete topology on \mathbb{C}^\times .

In his thesis, Tate associated to a quasi-character its *epsilon-factor* $\epsilon(\psi) \in \mathbb{C}^\times$, which enters the functional equation for the local L -function of ψ :

Notation 3.11. Composing with the local reciprocity map $F^\times \rightarrow G_F^{ab}$, consider ψ also as a character $F^\times \rightarrow \mathbb{C}^\times$. Define

$$\begin{aligned} n(\psi) &= \text{the conductor exponent of } \psi, \\ b(F) &= v_{\mathfrak{p}}(\Delta_{F/\mathbb{Q}_p}), \\ h &= \text{any elt. of } F^\times \text{ of valuation } -n(\psi) - b(F), \text{ e.g. } \pi_F^{-n(\psi)-b(F)}, \\ \epsilon(\psi) &= \begin{cases} \int_{h\mathcal{O}_F^\times} \psi(x^{-1}) e^{2\pi i \mathrm{Tr}_{F/\mathbb{Q}_p}(x)} dx & \text{for } \psi \text{ ramified,} \\ \int_{h\mathcal{O}_F^\times} \psi(h^{-1}) dx = \frac{\psi(h^{-1})}{|h|_F} \int_{\mathcal{O}_F^\times} dx & \text{for } \psi \text{ unramified.} \end{cases} \end{aligned}$$

The integrals are in effect finite sums (with the number of terms growing with the conductor of ψ and the ramification of F over \mathbb{Q}_p), so they can be explicitly computed for a given quasi-character.

Tate's theory of signs in the functional equations extends uniquely from quasi-characters to arbitrary Weil-Deligne representations

$$\rho : G_F \longrightarrow \mathrm{GL}_n(\mathbb{C}),$$

for all finite $F/K_{\mathfrak{p}}$, and all n :

Theorem 3.12 (Langlands-Deligne [8]). *There is a unique way to associate to each ρ its epsilon-factor $\epsilon(\rho) \in \mathbb{C}^\times$ such that*

- (1) *(Multiplicativity.) $\epsilon(\rho_1 \oplus \rho_2) = \epsilon(\rho_1)\epsilon(\rho_2)$.*
- (2) *(Inductivity in degree 0.) If $\rho_1, \rho_2 : G_F \longrightarrow \mathrm{GL}_n(\mathbb{C})$ have the same degree, then⁴*

$$\frac{\epsilon(\rho_1)}{\epsilon(\rho_2)} = \frac{\epsilon(\mathrm{Ind}_{G_F}^{G_{K_{\mathfrak{p}}}} \rho_1)}{\epsilon(\mathrm{Ind}_{G_F}^{G_{K_{\mathfrak{p}}}} \rho_2)}.$$

- (3) *(Quasi-characters.) For quasi-characters $\psi : G_F \rightarrow \mathbb{C}^\times$ the $\epsilon(\psi)$ are as in 3.11.*
- (4) *(Semi-simplification.) Writing ρ^{ss} for the semi-simplification of ρ ,*

$$\epsilon(\rho) = \epsilon(\rho^{ss}) \frac{\det(-\mathrm{Frob}_{\mathfrak{p}} | (\rho^{ss})^{I_{\mathfrak{p}}})}{\det(-\mathrm{Frob}_{\mathfrak{p}} | \rho^{I_{\mathfrak{p}}})}.$$

Remark 3.13. Uniqueness follows from the ‘Brauer induction’ argument: every semisimple Weil-Deligne representation is a \mathbb{Z} -linear combination of inductions of quasi-characters. Existence is harder: one has to understand ‘monomial relations’ between inductions and use Stickelberger’s theorem to prove that the ϵ -factors satisfy those relations.

Definition 3.14. Write $\mathrm{sgn} z = z/|z|$ for the ‘sign’ of $z \in \mathbb{C}^\times$ on the complex unit circle. The *local root number* of ρ is defined as

$$w(\rho) = \mathrm{sgn} \epsilon(\rho) = \frac{\epsilon(\rho)}{|\epsilon(\rho)|}.$$

Example 3.15. For $\rho = \psi$ a 1-dimensional unramified quasi-character,

$$w(\psi) = \mathrm{sgn} \psi(h^{-1}) = \mathrm{sgn} \psi(\mathrm{Frob}_{\mathfrak{p}})^{b(F)}.$$

In particular, the trivial representation has $w(\mathbf{1}) = 1$.

Example 3.16. Writing q for the size of the residue field of F , we find that the cyclotomic character also has

$$w(\chi) = (\mathrm{sgn} q)^{\cdots} = 1.$$

Because the ϵ -factors are multiplicative in direct sums and inductive in degree 0, clearly so are the root numbers; similarly,

$$w(\rho) = w(\rho^{ss}) \frac{\mathrm{sgn} \det(-\mathrm{Frob}_{\mathfrak{p}} | (\rho^{ss})^{I_{\mathfrak{p}}})}{\mathrm{sgn} \det(-\mathrm{Frob}_{\mathfrak{p}} | (\rho)^{I_{\mathfrak{p}}})}$$

as well. Here are some additional properties that are not hard to deduce:

⁴equivalently, $\epsilon(W) = \epsilon(\mathrm{Ind} W)$ for any virtual representation W of degree 0.

Proposition 3.17 (Tate [61], Deligne [8]).

- $w(\rho \oplus \rho^*) = (\det \rho)(-1)$, i.e. the image of -1 under

$$F^\times \xrightarrow{\text{loc. recip.}} G_F^{ab} \xrightarrow{\det \rho} \mathbb{C}^\times.$$

- $w(\rho_1 \otimes \rho_2) = w(\rho_1)^{\dim \rho_2} \cdot \text{sgn}(\det \rho_2)(\pi_F^{n(\rho_1)+b(F)\dim \rho_1})$ if ρ_2 is unramified.

3.4. Root numbers of elliptic curves.

Definition 3.18. Let $E/K_{\mathfrak{p}}$ be an elliptic curve. Define its *local root number*

$$w(E/K_{\mathfrak{p}}) = w(\rho); \quad \rho = (V_l E)^* \otimes_{\mathbb{Q}_l} \mathbb{C}.$$

For elliptic curves (and, generally, for abelian varieties) this is known to be independent of l (with $\mathfrak{p} \nmid l$) and of the embedding $\mathbb{Q}_l \hookrightarrow \mathbb{C}$, and it equals ± 1 .

Definition 3.19. The *global root number* of an elliptic curve E defined over a number field K is the product of the local root numbers over all places of K ,

$$w(E/K) = \prod_v w(E/K_v).$$

The product is finite since $w(E/K_v) = 1$ for primes of good reduction (see below). Also, we let $w(E/K_v) = -1$ for all Archimedean v .

Example 3.20 (Good reduction). If $E/K_{\mathfrak{p}}$ has good reduction, then ρ is unramified by the Néron-Ogg-Shafarevich criterion. Since $\det \rho$ is the cyclotomic character,

$$w(E/K_{\mathfrak{p}}) = w(1 \otimes \rho) = w(\mathbf{1})^2 \text{sgn} \det \rho(\pi_F^{\cdots}) = \text{sgn}(q^{\cdots}) = +1.$$

Example 3.21 (Split multiplicative reduction). If $E/K_{\mathfrak{p}}$ has split multiplicative reduction, then

$$V_l E = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}, \quad \rho = \begin{pmatrix} \chi^{-1} & 0 \\ * & 1 \end{pmatrix}, \quad \rho^{ss} = \begin{pmatrix} \chi^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

and $\det \rho = \chi$ as before. Applying the semi-simplification formula, we find

$$w(\rho) = w\left(\begin{pmatrix} \chi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot \frac{\text{sgn} \det(-\text{Frob}_{\mathfrak{p}} | \begin{pmatrix} \chi^{-1} & 0 \\ 0 & 1 \end{pmatrix})}{\text{sgn} \det(-\text{Frob}_{\mathfrak{p}} | 1)} = \frac{\text{sgn} \det \begin{pmatrix} -q^{-1} & 0 \\ 0 & 1 \end{pmatrix}}{\text{sgn} \det(1)} = \frac{1}{-1} = -1.$$

Example 3.22 (Non-split multiplicative reduction). If $E/K_{\mathfrak{p}}$ has non-split multiplicative reduction, the same computation shows that $w(E/K_{\mathfrak{p}}) = +1$.

To compute the root numbers of elliptic curves in the additive reduction case one has to understand ρ well enough. One case when this works well is when E admits an isogeny, forcing the Galois action on $V_l E$ to be less complicated than in general:

3.5. Root numbers of elliptic curves with an l -isogeny. As before, let $E/K_{\mathfrak{p}}$ be an elliptic curve, and suppose $\mathfrak{p} \nmid l$. Assume that there is a degree l isogeny defined over $K_{\mathfrak{p}}$,

$$\phi : E \longrightarrow E'.$$

Recall that ϕ is a non-constant morphism of curves mapping O to O , and over $\overline{K_{\mathfrak{p}}}$ it is an l -to-1 map; such a map automatically preserves addition on E , and therefore maps $E[l^n]$ to $E'[l^n]$, and hence $T_l E$ to $T_l E'$ as well.

Notation 3.23. Write $\ker \phi$ for the abelian group of points in the kernel of ϕ in $E(\overline{K_{\mathfrak{p}}})$; thus $\ker \phi \cong C_l$ as an abelian group. Write also

$$F := K_{\mathfrak{p}}(\text{coordinates of points in } \ker \phi) \subset K_{\mathfrak{p}}(E[l]).$$

The points in $\ker \phi$ are permuted by $G_{K_{\mathfrak{p}}}$, so $F/K_{\mathfrak{p}}$ is a Galois extension. This also means that the action of $G_{K_{\mathfrak{p}}}$ on $E[l]$ is reducible, that is of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. The Galois group $\text{Gal}(F/K_{\mathfrak{p}})$ is the image of

$$G_{K_{\mathfrak{p}}} \xrightarrow{\text{action on } E[l]} \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \xrightarrow{\text{top left corner}} \mathbb{F}_l^\times,$$

so it is cyclic of order dividing $l - 1$.

Notation 3.24. Write $(-1, F/K_{\mathfrak{p}})$ for the Artin symbol

$$(-1, F/K_{\mathfrak{p}}) = \begin{cases} +1, & \text{if } -1 \text{ is a norm from } F \text{ to } K_{\mathfrak{p}} \\ -1, & \text{otherwise.} \end{cases}$$

The Artin symbol is the main class-field-theoretic invariant for cyclic extensions, and the local root number of an elliptic curve with an isogeny turns out to be closely related to it:

Theorem 3.25. *If $E/K_{\mathfrak{p}}$ has additive reduction, $\mathfrak{p} \nmid l$ and $l \geq 5$, then*

$$w(E/K_{\mathfrak{p}}) = (-1, F/K_{\mathfrak{p}}).$$

Proof. We leave the potentially multiplicative case to the reader, and do the (slightly more involved) case of potentially good reduction.

Claim 1: $E[l]$ is unramified over F . Proof: Because $\det \rho = \chi$ and G_F acts trivially on $\ker \phi$, the image of G_F in $\text{Aut}(E[l]) = \text{GL}_2(\mathbb{F}_l)$ is contained in $\begin{pmatrix} 1 & * \\ 0 & \chi \end{pmatrix}$. As χ is unramified, the image I of inertia I_F is inside $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, so $|I|$ divides l . But $|I|$ divides 24 in the potentially good case, so I is trivial and $E[l]$ is unramified over F .

Claim 2: $E[l^n]$ is unramified over F for all $n \geq 1$. Proof: The image I_n of I_F in $\text{GL}_2(\mathbb{Z}/l^n\mathbb{Z})$ is of order dividing 24, as before. But the kernel of $\text{GL}_2(\mathbb{Z}/l^n\mathbb{Z}) \rightarrow \text{GL}_2(\mathbb{Z}/l\mathbb{Z})$ is of l -power order, so this map is injective on I_n . But by step 1, the image is trivial, so I_n is trivial as well.

Claim 3: E/F has good reduction. Proof: Néron-Ogg-Shafarevich.

Claim 4: The action of $G_{K_{\mathfrak{p}}}$ on $V_l E$ is abelian. Proof: We want to show that the commutator subgroup $G_{K_{\mathfrak{p}}}'$ acts trivially on $V_l E$. Now,

- $G_{K_{\mathfrak{p}}} \subset I_{K_{\mathfrak{p}}}$ as $G_{K_{\mathfrak{p}}}/I_{K_{\mathfrak{p}}} \cong \text{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}$ is abelian,

• The image of G_{K_p}' in $\text{Gal}(F/K_p)$ is trivial as $\text{Gal}(F/K_p)$ is abelian, so $G_{K_p}' \subset I_F$. But I_F acts trivially on $V_l E$ by Claim 2, as asserted.

Step 5. The local root number. A semisimple abelian action can be diagonalised over \mathbb{C} , so $\rho = \begin{pmatrix} \psi & 0 \\ 0 & \chi\psi^{-1} \end{pmatrix}$ for some quasi-character ψ . Now we can compute

$$\begin{aligned} w(\rho) &= w(\psi)w(\chi^{-1}\psi) = w(\psi)w(\psi^*) && \text{(unramified twist formula)} \\ &= w(\psi \oplus \psi^*) = \psi(-1) && (\rho \oplus \rho^* \text{ formula}) \\ &= \tilde{\psi}(-1) && \text{for any primitive character } \tilde{\psi} \\ &&& \text{of } \text{Gal}(F/K_p) \text{ that agrees} \\ &&& \text{with } \psi \text{ on inertia} \\ &= (-1, F/K_p). && \text{(local class field theory)} \end{aligned}$$

□

Remark 3.26. The theorem illustrates the fact that when G_{K_p} acts on $V_l E$ through an abelian quotient, we have enough formulae to determine the local root number. When the action is not abelian, the following result is very useful:

Theorem 3.27 (Fröhlich-Queyrut). *Suppose $F(\sqrt{\xi})/F$ is quadratic extension, and $\psi : G_{F(\sqrt{\xi})} \rightarrow \mathbb{C}^\times$ a quasi-character such that $\psi|_{F^\times} = 1$. Then*

$$w(\psi) = \psi(\xi) \in \{\pm 1\}.$$

Via the induction formula, the theorem computes the local root number of the Galois representation $w(\text{Ind } \psi)$. This is a 2-dimensional representation which is not necessarily abelian, e.g. the Galois image may be dihedral or quaternion (it has a cyclic subgroup of index 2). This is enough to determine the local root numbers of all elliptic curves over fields of residue characteristic at least 5 (i.e. $\mathfrak{p} \nmid 2, 3$); see [48].

In residue characteristics 2 and 3 the situation is a bit more complicated; see [26] for \mathbb{Q}_2 and \mathbb{Q}_3 , [32] for $\mathfrak{p}|3$ and [15, 65] for $\mathfrak{p}|2$, and also [16] for a general root number formula; see also [48, 49] for the root numbers of twists of elliptic curves.

4. PARITY OVER TOTALLY REAL FIELDS

We refer to Wintenberger [66] for the definition of modularity for elliptic curves over totally real fields. The two propositions below are essentially due to Taylor (see [62] proof of Thm. 2.4 and [63] proof of Cor. 2.2).

Proposition 4.1. *Let F/K be a cyclic extension of totally real fields, and E/K an elliptic curve. If E/F is modular, then so is E/K .*

Proof. Let π be the cuspidal automorphic representation associated to E/F . Pick a generator σ of $\text{Gal}(F/K)$. Then $\pi^\sigma = \pi$ and therefore by Langlands' base change theorem π descends to a cuspidal automorphic representation Π over K . Associated to Π there is a compatible system of λ -adic representations $\rho_{\Pi, \lambda}$, with λ varying over the primes of some number field k (see

[66]). Fix a prime λ of k , let l be the rational prime below it and write $V_l(E/F)$ for the Tate module of E/F tensored with \mathbb{Q}_l . The restriction of $\rho = \rho_{\Pi, \lambda}$ to $\text{Gal}(\bar{F}/F)$ agrees with $V = V_l(E/F) \otimes_{\mathbb{Q}_l} k_\lambda$. Because E cannot have complex multiplication over F (it is totally real), $V_l(E/F)$ is absolutely irreducible, and therefore the only representations that restrict to V are $W = V_l(E/K) \otimes_{\mathbb{Q}_l} k_\lambda$ and its twists by characters of $\text{Gal}(F/K)$. Hence ρ and W differ by a 1-dimensional twist, whence W is also automorphic and E/K is modular. \square

Proposition 4.2. *Let E be an elliptic curve over a totally real field K , and p a prime number. If the p -parity conjecture for E is true over every totally real extension of K where E is modular, then it is true for E/K .*

Proof. Because E is potentially modular (see [66], Thm. 1), there is a Galois totally real extension F/K over which E becomes modular. By Solomon's induction theorem, there are soluble subgroups $H_i < G = \text{Gal}(F/K)$ and integers n_i , such that the trivial character $\mathbf{1}_G$ can be written as

$$\mathbf{1}_G = \sum_i n_i \text{Ind}_{H_i}^G \mathbf{1}_{H_i}.$$

Write K_i for F^{H_i} . Since $\text{Gal}(F/K_i) = H_i$ is soluble, a repeated application of Proposition 4.1 shows that E/K_i is modular. By Artin formalism for L -functions,

$$L(E/K, s) = \prod_i L(E/K_i, s)^{n_i}.$$

On the other hand, writing $\mathcal{X} = \mathcal{X}_p(E/F)$, for every $H < G$ we have

$$\text{rk}_p E/K^H = \dim \mathcal{X}^H = \langle \mathcal{X}, \mathbf{1}_H \rangle_H = \langle \mathcal{X}, \text{Ind}_H^G \mathbf{1}_H \rangle_G.$$

The first equality is (2.2) and the last one is Frobenius reciprocity.

By assumption, the p -parity conjecture holds for E/K_i . Therefore

$$\begin{aligned} \text{rk}_p E/K &= \langle \mathcal{X}, \mathbf{1}_G \rangle_G = \sum_i n_i \text{rk}_p E/K_i \\ &\equiv \sum_i n_i \text{ord}_{s=1} L(E/K_i, s) = \text{ord}_{s=1} L(E/K, s) \pmod{2}. \end{aligned}$$

\square

Remark 4.3. The proof also shows that $L(E/K, s)$ has a meromorphic continuation to \mathbb{C} , with the expected functional equation. This is the same argument as in [63], proof of Cor. 2.2.

Theorem 4.4 ([12, 42, 16]). *Let K be a totally real field, and E/K an elliptic curve with non-integral j -invariant. Then the p -parity conjecture holds for E/K for every prime p .*

Proof. Let \mathfrak{p} be a prime of K with $\text{ord}_{\mathfrak{p}} j(E) < 0$. If E has additive reduction at \mathfrak{p} , it becomes multiplicative over some totally real quadratic extension

$K(\sqrt{\alpha})$, and the quadratic twist E_α/K has multiplicative reduction at \mathfrak{p} as well. Because

$$(4.5) \quad \begin{aligned} w(E/K(\sqrt{\alpha})) &= w(E/K)w(E_\alpha/K) & \text{and} \\ \text{rk}_p E/K(\sqrt{\alpha}) &= \text{rk}_p E/K + \text{rk}_p E_\alpha/K, \end{aligned}$$

it suffices to prove the theorem for elliptic curves with a prime of multiplicative reduction. Since multiplicative reduction remains multiplicative in all extensions, by Proposition 4.2 we may also assume that E is modular.

By Friedberg-Hoffstein's theorem [21] Thm. B, there is a quadratic extension $M = K(\sqrt{\beta})$ of K with a prescribed behaviour at a given finite set of primes of K , such that the quadratic twist E_β has analytic rank ≤ 1 . If we require that the multiplicative primes are unramified, then E_β also has a prime of multiplicative reduction. By Zhang's theorem ([67] Thm. A) that generalises the Gross-Zagier formula over the rationals, $\text{III}_{E_\beta/K}$ is finite and the Mordell-Weil rank of E_β/K agrees with its analytic rank; in particular, the p -parity conjecture holds for E_β/K . To prove it for E/K we just need to show it for E/M (cf. (4.5)).

Suppose $p = 2$. Fix an invariant differential ω for E/K , and choose M so that all infinite places, all bad places for E , and ones where ω is not minimal are split in M/K . By (2.20) in §2.6,

$$\text{rk}_2 E/K(\sqrt{\beta}) \equiv \text{ord}_2 \frac{C_{E/K(\sqrt{\beta})} C_{E_\beta/K}}{C_{E/K}} \equiv 0 \pmod{2},$$

since the other finite primes do not contribute to C . For the same reason $w(E/M) = (\pm 1)^2 = 1$; see §8 where this is spelled out in more detail, or apply Theorem 2.21. So the 2-parity holds for E/M , as required.

Finally suppose $p > 2$. Then choose M so that (a) all bad places for E except one multiplicative place \mathfrak{p} are split, (b) \mathfrak{p} is inert, and (c) all real places are inert, i.e. M is totally complex. Let $F_\infty/M/K$ be the p -adic anticyclotomic tower; thus F_∞ is the largest Galois extension of K containing M such that $G = \text{Gal}(F_\infty/K)$ is of the form $\mathbb{Z}_p^r \rtimes C_2$ with C_2 acting by -1 . The Artin representations of G are $\mathbf{1}$ and ϵ that factor through $\text{Gal}(M/K)$, plus two-dimensional ones of the form $\rho = \text{Ind}_M^K \chi$ for non-trivial $\chi : \mathbb{Z}_p^r \rightarrow \mathbb{C}^\times$. Such a χ factors through a dihedral extension F/K in F_∞ , and by §2.5 we have

$$\langle \mathbf{1} + \epsilon + \rho, \mathcal{X}_p(E/F) \rangle \equiv \text{ord}_p \frac{C_{E/F}}{C_{E/M}} \equiv 1 \pmod{2},$$

where the last equality again uses the fact that all bad primes for E and all infinite primes split into pairs in M (and therefore in F as well), except for \mathfrak{p} which gives the '1'. For the same reason, we have

$$w(E/K)w(E/K, \epsilon) = w(E/M, \chi) = -1.$$

We already know the p -parity conjecture for E_β ,

$$w(E/K, \epsilon) = w(E_\beta/K) = (-1)^{\text{ord}_2 \text{rk}_2 E_\beta/K} = (-1)^{\langle \epsilon, X_p(E/F) \rangle},$$

so to show the p -parity conjecture for the trivial twist, we just need to verify it for a *single* non-trivial χ . Precisely such a character is provided by the anticyclotomic theory: a combination of Cornut–Vatsal’s [7] Thm. 4.2 with Nekovář’s [40] Thm. 3.2 gives a χ such that the χ -component of $\mathcal{X}_p(E/F)$ has multiplicity 1, which is an odd number, as required. \square

Remark 4.6. The same idea lies behind the proof of the p -parity conjecture over \mathbb{Q} , see [39] ($p = 2$) and [12] (odd p) for details.

5. THE 2-ISOGENY THEOREM

Theorem 5.1. *Let K be a number field and E/K an elliptic curve with $E(K)[2] \neq 0$. Then*

$$(-1)^{\text{rk}_2 E/K} = w(E/K).$$

In particular, finiteness of III implies the parity conjecture for E .

The proof will occupy the whole of §5.

Notation 5.2. Fix a 2-torsion point $0 \neq P \in E(K)[2]$, and let $\phi : E \rightarrow E'$ be a 2-isogeny whose kernel is $\{O, P\}$. Furthermore, translate P to $(0, 0)$, so that E and E' become

$$(E_{a,b} =) \quad \begin{array}{ll} E : & y^2 = x^3 + ax^2 + bx, \\ E' : & y^2 = x^3 - 2ax^2 + \delta x, \end{array} \quad \begin{array}{l} a, b \in \mathcal{O}_K, \\ \delta = a^2 - 4b, \end{array}$$

with $\phi : E \rightarrow E'$ given by

$$\phi : (x, y) \mapsto (x + a + bx^{-1}, y - bx^{-2}y).$$

Notation 5.3. Denote

$$\sigma_\phi(E/K_v) = (-1)^{\text{ord}_2 \frac{|\text{coker } \phi_v|}{|\ker \phi_v|}} = (-1)^{1 + \text{ord}_2 |\text{coker } \phi_v|},$$

where ϕ_v is the induced map on local points $\phi_v : E(K_v) \rightarrow E'(K_v)$. (Note that $\ker \phi_v = \{O, (0, 0)\}$ is always of size 2.)

Strategy. Recall from Theorem 2.8 and the definition of the global root number 3.19 that

$$(5.4) \quad (-1)^{\text{rk}_2 E/K} = \prod_v \sigma_\phi(E/K_v); \quad \prod_v w(E/K_v) = w(E/K),$$

so a natural strategy is to make a term-by-term comparison at all places.

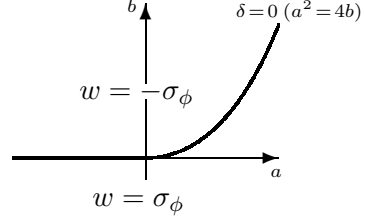
Write $F = K_v$ for some completion of K .

5.1. Complex places. Suppose $F = \mathbb{C}$. Because it is algebraically closed, $\phi_v : E(\mathbb{C}) \rightarrow E'(\mathbb{C})$ is surjective, that is $|\text{coker } \phi_v| = 1$. Thus,

$$w(E/F) = -1 = \sigma_\phi(E/F).$$

5.2. Real places.

Suppose $F = \mathbb{R}$. It turns out, somewhat surprisingly, that $w(E/\mathbb{R})$ and $\sigma_\phi(E/\mathbb{R})$ may not be equal: $w(E/F) = -1$ as in the complex case but $\phi_v : E(\mathbb{R}) \rightarrow E'(\mathbb{R})$ is not always surjective. For varying $a, b \in \mathbb{R}$ the picture is given on the right.



To see this, consider the structure of the group of real components for E and E' ; recall that the group of real points is connected if and only if the discriminant of the equation is negative:

- If $(-2a)^2 - 4\delta = 16b < 0$, then $E'(\mathbb{R}) \cong S^1$, so ϕ_v is surjective.
- If $b > 0$ and $\delta < 0$ then $E(\mathbb{R}) \cong S^1$ and $E'(\mathbb{R}) \cong S^1 \times \mathbb{Z}/2\mathbb{Z}$, so ϕ_v is not surjective.
- If $b, \delta > 0$, then $E(\mathbb{R}) \cong S^1 \times \mathbb{Z}/2\mathbb{Z} \cong E'(\mathbb{R})$. Here ϕ_v is surjective if and only if the points $\mathcal{O}, (0, 0)$ of $\ker \phi$ lie on the same connected component. (If they are on different components, the image of ϕ is connected; otherwise, the identity component of $E(\mathbb{R})$ maps 2-to-1 to the identity component of $E'(\mathbb{R})$, so the other component maps to the other component since $\deg \phi = 2$.) So ϕ_v is surjective if and only if 0 is the rightmost root of $x^3 + ax^2 + bx$, and this is equivalent to a being positive.

5.3. The correction term. To save our strategy, which seems to be somewhat in ruins, we have to introduce a correction term that measures the difference between $w(E/F)$ and $\sigma_\phi(E/F)$. It should be trivial at complex places and depend on the signs of a, b and δ at real places, so it is natural to consider something like

$$(a, -b)_F (-a, \delta)_F$$

where $(x, y)_F$ is the Hilbert symbol:

Definition 5.5. Let F be a local field of characteristic zero. For $x, y \in F^\times$ the Hilbert symbol $(x, y)_F = \pm 1$ is

$$(x, y) = (x, F(\sqrt{y})/F) = \begin{cases} +1, & \text{if } x \text{ is a norm from } F(\sqrt{y}) \text{ to } F, \\ -1, & \text{otherwise.} \end{cases}$$

Recall that (x, y) is symmetric, bilinear as a map $F^\times \times F^\times \rightarrow \{\pm 1\}$, and satisfies $(1 - x, x) = 1$ for $x \neq 0, 1$. In a field of residue characteristic $\neq 2$ it is explicitly determined by

$$\begin{aligned} F = \mathbb{C} : & \quad (x, y) = 1 \text{ always;} \\ F = \mathbb{R} : & \quad (x, y) = -1 \Leftrightarrow x, y < 0; \\ F/\mathbb{Q}_p \text{ finite, } p \neq 2 : & \quad (\text{unit}, \text{unit}) = 1, \\ & \quad (\text{uniformiser}, \text{non-square unit}) = -1. \end{aligned}$$

The suggested correction term $(a, -b)_F (-a, \delta)_F$ is trivial at complex places and it is precisely set up to give the right signs at the real places. A few experiments suggest, that up to a missing factor of 2 in front of $-a$ (invisible

over the reals), this is the right correction at *all* completions $F = K_v$, not just at infinite places. One technical problem is that the Hilbert symbols do not make sense when $a = 0$, but this is easy to fix: if $|a|_F$ is small, then

$$\begin{aligned} (a, -b)_F(-2a, a^2 - 4b)_F &= (a, -b)_F(-2a, 1 - \frac{a^2}{4b})_F(-2a, -4b)_F \\ &= (a, -b)_F(-2a, \square)_F(a, -4b)_F(-2, -4b)_F \\ &= (-2, -b)_F, \end{aligned}$$

because elements of F close to 1 are squares. So we may state

Conjecture 5.6. *Let F be a local field of characteristic zero, and E/F an elliptic curve with a 2-isogeny $\phi : E \rightarrow E'$ over F . In the notation of 5.2,*

$$w(E/F) = \sigma_\phi(E/F) \cdot \begin{cases} (a, -b)_F(-2a, a^2 - 4b)_F, & \text{if } a \neq 0 \\ (-2, -b)_F, & \text{if } a = 0. \end{cases}$$

This conjecture together with (5.4) implies Theorem 5.1. Indeed, if E as in 5.2 is now defined over a number field K , the product formula for the Hilbert symbol

$$\prod_v (x, y)_{K_v} = 1, \quad \text{for all } x, y \in K^\times$$

implies that the correction term disappears globally.

Next, we prove the conjecture in a few cases. As we dealt with infinite places already, assume from now on that $F = K_v$ is non-Archimedean, i.e. a finite extension of \mathbb{Q}_p . Also we may deal with the annoying exceptional case $a = 0$: if we prove the conjecture in all cases when $|a|_F$ is small, then it holds when $a = 0$ as well, because both the left- and the right-hand side in the conjecture are continuous functions of the coefficients a and b of E ; we already proved this for the Hilbert symbols, and for $C(E/F, \omega)$ and $w(E/F)$ this is a general fact:

Proposition 5.7. *Suppose E/F an elliptic curve in Weierstrass form,*

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in F.$$

There is an $\epsilon > 0$ such that changing the a_i to any a'_i with $|a_i - a'_i|_F < \epsilon$ does not change the conductor, minimal discriminant, Tamagawa number, $C(E, \frac{dx}{2y+a_1x+a_3})$, the root number of E and the Tate module $T_l E$ as a $\text{Gal}(\bar{F}/F)$ -module for any given $l \neq p$.

Proof. The assertion for the conductor, minimal discriminant, Tamagawa number and C follows from Tate's algorithm ([60] or [55] IV.9). The claim for $T_l E$ is a result of Kisin ([29] p. 569), and the root number is a function of $V_l E = T_l E \otimes \mathbb{Q}_l$. Alternatively, that the root number is locally constant can be proved in a more elementary way: see Helfgott's [27] Prop. 4.2 when E has potentially good reduction; in the potentially multiplicative case it follows from Rohrlich's formula ([48] Thm. 2(ii)). \square

So assume from now on that $a \neq 0$. Note also that our chosen model $y^2 = x^3 + ax^2 + bx$ is unique up to transformations $a \mapsto u^2a, b \mapsto u^4b$ for

$u \in K^\times$. As these do not change the Hilbert symbols $(a, -b)$, $(-2a, a^2 - 4b)$ and $(-2, -b)$, we may and will assume that the model is integral and minimal when $v \nmid 2$. Now we prove the conjecture in all cases when E is semistable with $v \nmid 2$; recall that $\text{ord}_2 C_{E/F} = \text{ord}_2 c(E/F)$ in this case, since the quotient C/c is a power of the residue characteristic.

5.4. Good reduction at $v \nmid 2$. Here $w(E/F) = 1$, $\sigma_\phi(E/F) = 1$ and $b, \delta \in \mathcal{O}_F^\times$. If $a \in \mathcal{O}_F^\times$, then both the Hilbert symbols are (unit, unit), hence trivial. For $a \equiv 0 \pmod{m_F}$, the expression $-b\delta \equiv 4b^2 \pmod{m_F}$ is a non-zero square mod m_F , so the product of the Hilbert symbols is again trivial.

5.5. Split multiplicative reduction at $v \nmid 2$. Write E/F as a Tate curve ([55] §V.3)

$$E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q), \quad E(F) \cong F^\times / q^\mathbb{Z},$$

with $q \in m_F$ of valuation $v(q) = v(\Delta)$. The coefficients have expansions

$$a_4(q) = -5s_3(q), \quad a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12}, \quad s_k(q) = \sum_{n \geq 1} \frac{n^k q^n}{1 - q^n},$$

and they start

$$\begin{aligned} a_4(q) &= -5q - 45q^2 - 140q^3 - 365q^4 - 630q^5 + O(q^6), \\ a_6(q) &= -q - 23q^2 - 154q^3 - 647q^4 - 1876q^5 + O(q^6). \end{aligned}$$

The two-torsion, as a Galois set, is $\{1, -1, \sqrt{q}, -\sqrt{q}\}$. For $u \neq 1$ in this set, the corresponding point on E has coordinates

$$\begin{aligned} X(u, q) &= \frac{u}{(1-u)^2} + \sum_{n \geq 1} \left(\frac{q^n u}{(1-q^n u)^2} + \frac{q^n u^{-1}}{(1-q^n u^{-1})^2} - 2 \frac{q^n}{(1-q^n)^2} \right), \\ Y(u, q) &= \frac{u^2}{(1-u)^3} + \sum_{n \geq 1} \left(\frac{q^{2n} u^2}{(1-q^n u)^3} + \frac{q^n u^{-1}}{(1-q^n u^{-1})^2} + \frac{q^n}{(1-q^n)^2} \right). \end{aligned}$$

We now have two cases to consider: the 2-torsion point $(X(-1, q), Y(-1, q)) \in E_q$ and (renaming $\pm\sqrt{q}$ by q) the 2-torsion point $(X(q, q^2), Y(q, q^2)) \in E_{q^2}$. In both cases, we have $c(E)/c(E') = 2^{\pm 1}$ and $w(E/F) = -1$, so we need

$$(5.8) \quad (a, -b)(-2a, \delta) = 1,$$

where a, b, δ are the invariants of the curve transformed back to the form 5.2 with the 2-torsion point at $(0, 0)$. First of all, E_q has a model

$$y^2 = x^3 + x^2/4 + a_4(q)x + a_6(q).$$

Let $r = -X(u, q)$, and write $a_4 = a_4(q)$, $a_6 = a_6(q)$. Then, after translation, the curve becomes

$$E : y^2 = x^3 + ax^2 + bx, \quad a = 1/4 - 3r, \quad b = 2a_4 - r/2 + 3r^2.$$

Suppose we are in Case 1, so $r = -X(-1, q)$. Then the substitution

$$(5.9) \quad x \rightarrow 4x - 2r + 1/2, \quad y \rightarrow 8y + 4x$$

transforms E' into the form

$$E^\dagger : y^2 + xy = x^3 + (-5q^2 + O(q^4))x + (-q^2 + O(q^4)) .$$

We use the notation $O(q^n)$ to indicate a power series in q with coefficients in \mathcal{O}_F that begins with $a_n q^n + \dots$. In fact, $E^\dagger = E_{q^2}$ but we will not need this; it is only important that it is again a Tate curve (in particular, this model is minimal). From the expansions

$$r = 1/4 + 4O(q), \quad a = -1/2 + 4O(q), \quad b = 1/16 + O(q),$$

we have

$$(a, -b) = (\text{unit}, \text{unit}) = 1, \quad (-2a, \delta) = (1 - 8O(q), \delta) = (\square, \delta) = 1 .$$

Case 2 is similar: here

$$a = 1/4 + 2O(q), \quad b = q + O(q^2), \quad \delta = 1/16 + O(q) ,$$

so a and δ are squares in F , and both Hilbert symbols are therefore trivial.

5.6. Nonsplit multiplicative reduction at $v \nmid 2$. Let $\eta \in \mathcal{O}_F^\times$ be a non-square unit, so that $F(\eta)/F$ is the unramified quadratic extension of F . Consider the twist of E by η ,

$$\begin{aligned} E : \quad y^2 &= x^3 + ax^2 + bx , \\ E_\eta : \quad y^2 &= x^3 + \eta ax^2 + \eta^2 bx . \end{aligned}$$

It has split multiplicative reduction, so by (5.8)

$$(\eta a, -\eta^2 b)(2\eta a, \eta^2(a^2 - 4b)) = 1.$$

Now

$$\begin{aligned} (\eta a, -\eta^2 b) &= (\eta a, -b) = (\eta, -b)(a, -b), \\ (2\eta a, \eta^2(a^2 - 4b)) &= (2\eta a, a^2 - 4b) = (\eta, a^2 - 4b)(2a, a^2 - 4b), \end{aligned}$$

so comparing with the Hilbert symbols $(a, -b)$ and $(-2a, \delta)$ we have an extra term

$$(5.10) \quad (\eta, -b(a^2 - 4b)) = (\eta, -b^2 \Delta(E')/16\Delta(E)) = (\eta, -\Delta(E')/\Delta(E)) .$$

Because x is a norm from $F(\eta)^\times$ to F^\times if and only if $v(x)$ is even, this Hilbert symbol is trivial precisely when $v(\Delta(E')) \equiv v(\Delta(E)) \pmod{2}$. From Tate's algorithm ([55], IV.9.4, Step 2),

$$c(E) = \begin{cases} 1, & v(\Delta(E)) \text{ odd}, \\ 2, & v(\Delta(E)) \text{ even}, \end{cases} \quad c(E') = \begin{cases} 1, & v(\Delta(E')) \text{ odd}, \\ 2, & v(\Delta(E')) \text{ even}, \end{cases}$$

so the correction term (5.10) is trivial if and only if $c(E)/c(E')$ has even 2-valuation. This proves Conjecture 5.6 in the non-split multiplicative case, when $v \nmid 2$.

5.7. Deforming to totally real fields. There are several other reduction types, including all cases when $v \nmid 2$ and when E has good ordinary or multiplicative reduction at $v|2$, when Conjecture 5.6 can be shown to hold directly [11]. It would be satisfying to do this in the remaining cases as well:

Problem 5.11. *If F/\mathbb{Q}_2 is finite, and E/F has good supersingular or additive reduction, prove Conjecture 5.6 directly.*

Having tried and failed to do this, we will complete the proof of Conjecture 5.6 and of Theorem 5.1 by a global argument. The idea is that if E/K is an elliptic curve over a number field and the conjecture holds for it at all primes but one, the conjecture at the remaining prime is *equivalent* to the 2-parity conjecture for E . But there is a large supply of elliptic curves E , namely those defined over totally real fields with non-integral j -invariants, for which we know the 2-parity conjecture. There are more than enough of these to approximate any given elliptic curve over F .

Assumption 5.12. *From now on F/\mathbb{Q}_2 is finite, $a, b \in F$ satisfy $b \neq 0, a^2 - 4b \neq 0$, and $E_{a,b}$ is an elliptic curve as in 5.2,*

$$E_{a,b} : y^2 = x^3 + ax^2 + bx.$$

Lemma 5.13. *There exists a totally real field K with a unique place $v_0|2$, and $\tilde{a}, \tilde{b} \in K$ such that*

- (1) $K_{v_0} \cong F$.
- (2) *Under this identification, $|a - \tilde{a}|_{v_0}$ and $|b - \tilde{b}|_{v_0}$ are so small that all terms in Conjecture 5.6 are the same for $E_{a,b}/F$ and for $E_{\tilde{a},\tilde{b}}/K_{v_0}$.*
- (3) $E_{\tilde{a},\tilde{b}}$ *is semistable at all primes $v \neq v_0$ of K and has non-integral j -invariant.*

Proof. (1) Say $F = \mathbb{Q}_2[x]/(f)$ with monic $f \in \mathbb{Q}_2[x]$. If $\tilde{f} \in \mathbb{Q}[x]$ is monic and 2-adically close enough to f , it defines the same extension of \mathbb{Q}_2 by Krasner's lemma. Now pick such an \tilde{f} which is also \mathbb{R} -close to any polynomial in $\mathbb{R}[x]$ whose roots are real (weak approximation), and set $K = \mathbb{Q}[x]/(\tilde{f})$ and v_0 to be the prime above 2 in K .

(2) Next, provided $\tilde{a}, \tilde{b} \in K$ are close enough to a, b at v_0 , the continuity of the Hilbert symbol (and the computation preceding Conjecture 5.6) and Proposition 5.7 imply that (2) holds.

(3) As v_0 is the unique prime above 2, it suffices to guarantee that $E_{\tilde{a},\tilde{b}}$ is semistable at primes $v \nmid 2$ of K and has $j(E) \notin O_K$. The curve has standard invariants

$$c_4 = 16(\tilde{a}^2 - 3\tilde{b}), \quad c_6 = -32\tilde{a}(2\tilde{a}^2 - 9\tilde{b}), \quad \Delta = 16\tilde{b}^2(\tilde{a}^2 - 4\tilde{b}).$$

Pick any non-zero $\tilde{a} \in O_K$ which is close to a at v_0 . Next, choose any $\tilde{b} \in O_K$ which is close to b at v_0 and close to 1 at all primes $v \neq v_0$ that divide \tilde{a} . Then either Δ or c_4 is a unit at every prime $v \neq v_0$, and this ensures that E is semistable outside v_0 ([54] Prop. VII.5.1). If, in addition, we force b to

be divisible by at least one prime $\mathfrak{p} \nmid 2$ (weak approximation all the time), this guarantees that $j(E)$ is non-integral at \mathfrak{p} . \square

Claim. Conjecture 5.6 is true.

Proof. First we finish off the case when F has residue characteristic 2. Let F/\mathbb{Q}_2 and $E_{a,b}$ be as above, and K and $E = E_{\tilde{a},\tilde{b}}$ as in the lemma. We want to show that Conjecture 5.6 holds for $E_{a,b}/F$, equivalently for E/K_{v_0} .

Because K has a unique prime v_0 above 2, and E is semistable at all other primes, Conjecture 5.6 holds for E/K_v for all $v \neq v_0$. In view of (5.4), the conjecture at v_0 is equivalent to the 2-parity conjecture for E/K . But E has non-integral j -invariant and K is totally real, so the latter holds.

Finally, suppose F has odd residue characteristic p , and E/F has additive reduction. Exactly as in Lemma 5.13, find a totally real field K with a place $v_0|p$ and \tilde{E}/K which is v_0 -adically close to E , and semistable at all $v \neq v_0$. Again, the 2-parity conjecture for \tilde{E}/K together with Conjecture 5.6 at all $v \neq v_0$ (now including $v|2$) proves Conjecture 5.6 at v_0 as well. \square

As explained above, reversing the argument yet again proves Theorem 5.1.

6. THE p -ISOGENY CONJECTURE

What is the analogue of the results of the previous section for an isogeny whose degree is an odd prime p ? Let $\phi : E \rightarrow E'$ be such an isogeny, with E, E' and ϕ all defined over K . Again, write

$$\sigma_\phi(E/K_v) = (-1)^{\text{ord}_p \frac{|\text{coker } \phi_v|}{|\text{ker } \phi_v|}},$$

so that by Theorem 2.8,

$$(6.1) \quad (-1)^{\text{rk}_p E/K} = \prod_v \sigma_\phi(E/K_v); \quad \prod_v w(E/K_v) = w(E/K).$$

Again, let us make a term-by-term comparison at all places. Fix a place v of K . The difference between $w(E/K_v)$ and $\sigma_\phi(E/K_v)$ is suggested by the root number computation of Theorem 3.25, and we formulate the following

Conjecture 6.2. *Let \mathcal{K} be a local field of characteristic zero, and E/\mathcal{K} an elliptic curve with a p -isogeny $\phi : E \rightarrow E'$ over \mathcal{K} . Write $\mathcal{F} = \mathcal{K}(\text{ker } \phi)$. Then*

$$w(E/\mathcal{K}) = \sigma_\phi(E/\mathcal{K}) (-1, \mathcal{F}/\mathcal{K}).$$

If this conjecture is true, applying it for E/K at completions and taking the product over all places, we get the p -parity conjecture for E/K . Let us verify that the conjecture holds in most situations:

If $\mathcal{K} = \mathbb{C}$ then $|\text{ker } \phi| = p$, $|\text{coker } \phi| = 1$, $w(E/\mathcal{K}) = -1$ and $\mathcal{F} = \mathcal{K}$, so the formula holds. The same is true if $\mathcal{K} = \mathbb{R}$ and the points in $\text{ker } \phi$ are real.

If $\mathcal{K} = \mathbb{R}$ and the points in $\text{ker } \phi$ are not defined over \mathbb{R} , then then $|\text{ker } \phi| = 1$, $|\text{coker } \phi| = p$ and $\mathcal{F} = \mathbb{C}$, so the correction term $(-1, \mathbb{C}/\mathbb{R}) = -1$ compensates for the kernel, and the formula holds again.

Suppose \mathcal{K} is a finite extension of \mathbb{Q}_l for $l \neq p$. The standard exact sequences for $E_0(\mathcal{K}) \subset E(\mathcal{K})$ and $E_1(\mathcal{K}) \subset E_0(\mathcal{K})$ ([54] Ch. VII) show that

$$\sigma_\phi(E/\mathcal{K}) = (-1)^{\text{ord}_p \frac{c(E/\mathcal{K})}{c(E'/\mathcal{K})}},$$

where c is the local Tamagawa number. (Use that E and E' have the same reduction type and the same number of points over the residue field, and that $[p]$ is an isomorphism on the formal groups when $p \neq l$.) Now we have a few cases:

If E/\mathcal{K} has good reduction, then $\mathcal{K}(E[p])/\mathcal{K}$ is unramified by the Néron-Ogg-Shafarevich criterion, in particular \mathcal{F}/\mathcal{K} is unramified as well. So all units are norms in this extension by class field theory, in particular -1 is. The Tamagawa numbers are trivial as well, so all terms in the conjecture are 1.

If E/\mathcal{K} has multiplicative reduction, then \mathcal{F}/\mathcal{K} is still unramified, since the inertia in $\mathcal{K}(E[p])/\mathcal{K}$ is pro- p in the multiplicative case and $[\mathcal{F} : \mathcal{K}](p-1)$ is coprime to p ; so the Artin symbol is still 1. Now $w(E/\mathcal{K}) = -1$ as opposed to the good case, but also the quotient $\frac{c(E/\mathcal{K})}{c(E'/\mathcal{K})}$ is either p or $1/p$ (same argument as in §5.5 for $p = 2$), and the conjecture holds.

If E/\mathcal{K} has additive reduction and p, l are at least 5, then the Tamagawa numbers have order ≤ 4 , coprime to p . So the conjecture claims that $w(E/\mathcal{K}) = (-1, \mathcal{F}/\mathcal{K})$, and this is precisely what we proved in Theorem 3.25; in fact, it is this computation that suggests the correction term in the conjecture.

When $p = l$, the problem becomes trickier. It is still manageable when E/\mathcal{K} is semistable, by a careful analysis of σ_ϕ which also involves the action of ϕ on the formal groups (see [11]). For $p = 3$ the whole conjecture can be settled by a deformation argument, exactly as we did for $p = 2$ (see [16]; presumably the argument there also works when $p = 5$ and 7, when $X_0(p)$ still has genus 0). The remaining problem is

Problem 6.3. *Prove Conjecture 6.2 when $p > 7$, the field \mathcal{K} has residue characteristic p and E/\mathcal{K} has additive reduction.*

Coates, Fukaya, Kato and Sujatha [6] have proved this conjecture when E/\mathcal{K} acquires semistable reduction after an abelian extension of \mathcal{K} , with a ‘tour de force’ computation with crystalline cohomology. Moreover, they proved it for arbitrary principally polarised abelian varieties; p -isogeny has to be replaced by a p^g -isogeny with a totally isotropic kernel. It would be very interesting to find at least a conjectural generalisation for $p = 2$:

Problem 6.4. *Suppose \mathcal{K} is a local field of characteristic 0, and $\phi : A \rightarrow A'$ is an isogeny of principally polarised abelian varieties over K such that $\phi^t \phi = [2]$ (up to isomorphisms given by the polarisations). Find a relation between $\sigma_\phi(A/\mathcal{K})$ and $w(A/\mathcal{K})$.*

This would have important parity implications for abelian varieties. We also mention here that Trihan and Wuthrich [64] have proved an analogue

of Conjecture 6.2 over function fields of characteristic p , when ϕ is the dual of the Frobenius isogeny. Because this isogeny exists for elliptic curve over a function field, they thus proved the p -parity conjecture in characteristic p .

Finally, returning to Conjecture 6.2 itself, it would also be very interesting to see whether it can be settled by a deformation argument, like we did it for $p = 2$. The problem is that because $X_0(p)$ has positive genus for large p , it is unclear how to deform elliptic curves with a p -isogeny while keeping control over other places of the resulting totally real field. It is known how to construct points of essentially arbitrary varieties over totally real fields (see e.g. [46]), but it is not clear whether these results are sufficient in this case.

7. LOCAL COMPATIBILITY IN S_3 -EXTENSIONS

To complete the proof of Theorem A it remains to settle the second equality in Theorem 2.6. Thus, suppose E/K is an elliptic curve, F/K is a Galois S_3 -extension and M/K and L/K are the quadratic and cubic intermediate extensions. We want to show that

$$(\dagger_{\text{glo}}) \quad w(E/K)w(E/M)w(E/L) = (-1)^{\text{ord}_3 \frac{C_{E/F}C_{E/K}^2}{C_{E/M}C_{E/L}^2}}.$$

Both sides are products of local terms, and, as in the isogeny case, we want to compare the contributions above each place of K .

Fix such a place v of K , and denote by w_K, w_M, w_L and w_F the product of the local root numbers of E over the places of K, M, L and F above v . Fix also an invariant differential ω for E/K and write C_K, C_M etc. for the product of the modified local Tamagawa numbers of E over the places of K, M etc., computed using ω (see end of Notation 1.1). We need to show the following local statement:

$$(\dagger_{\text{loc}}) \quad w_K w_M w_L = (-1)^{\text{ord}_3 \frac{C_F C_K^2}{C_M C_L^2}}.$$

Lemma 7.1. *If there is more than one place above v in F or E has good reduction at v then (\dagger_{loc}) holds.*

Proof. It is easy to see that for $v|\infty$ both the left-hand side and the right-hand side are trivial⁵, so suppose v is a finite place. Then the right-hand side is the same as $\text{ord}_p(C_F/C_M) \bmod 2$. Now consider the following cases.

Case 1: v splits in M/K . (In particular, this must happen if F/M is inert above v .) As the number of primes above v in M is even, C_F and C_M are both squares, and $w_M = 1$. Since in this case L_v/K_v is Galois of odd degree, $w_L = w_K$ by Kramer–Tunnell [31], proof of Prop. 3.4.

Case 2: the prime above v in M splits in F/M . Then $C_F = C_M^3$, so C_F/C_M is a square. Under the action of the decomposition group D_v at v , the G -sets $G/\text{Gal}(F/L)$ and $(G/\text{Gal}(F/M)) \amalg (G/G)$ are isomorphic. So the number of primes above v with a given ramification and inertial degree

⁵This is true in any Brauer relation, see e.g [14] Proof of Cor. 3.4

is the same in L as in M plus in K . It follows that the local root numbers cancel, $w_K w_M w_L = 1$.

Case 3: F/M is ramified above v and E is semistable at v . The contributions from ω cancel modulo squares, and $w_K = w_L$. If E has split multiplicative reduction over a prime of M above v , this prime contributes p to C_F/C_M and -1 to the root number. If the reduction is either good or non-split, it contributes to neither. \square

Proposition 7.2. *The formula (\dagger_{loc}) holds in all cases.*

Proof. The only case not covered by the lemma above is when E/K has additive reduction at v and F/K has a unique prime \tilde{v} above v . We will use a continuity argument to settle this case.

Pick an S_3 -extension \mathcal{F}/\mathcal{K} of totally real number fields with completions $\mathcal{K}_w = K_v$ and $\mathcal{F}_{\tilde{w}} = F_{\tilde{v}}$ for some prime $\tilde{w}|w$ in \mathcal{F}/\mathcal{K} (same argument as in Proposition 5.7). Choose an elliptic curve \mathcal{E}/\mathcal{K} which is close enough w -adically to E/K_v , with semistable reduction at all places $\neq w$ where \mathcal{F}/\mathcal{K} is ramified and at least one prime of multiplicative reduction. By ‘close enough’ we mean that the left- and the right-hand sides of (\dagger_{loc}) are the same for E/K_v and $\mathcal{E}/\mathcal{K}_w$ (Proposition 5.7).

By the 3-parity conjecture for \mathcal{E} over the intermediate fields of \mathcal{F}/\mathcal{K} (Theorem 4.4) and Theorem 2.9, we find that (\dagger_{glo}) holds. Since the terms in it agree at all places except possible w by Lemma 7.1, they must agree at w as well. This proves (\dagger_{loc}) for $\mathcal{E}/\mathcal{K}_w$ and hence for E/K_v as well. \square

This completes the proof of Theorem A.

8. PARITY PREDICTIONS

The purpose of this final section is to collect some peculiar predictions of the parity conjecture concerning ranks of elliptic curves over number fields.

Definition 8.1. Write $H(\frac{p}{q}) = \max(|p|, |q|)$ for the usual ‘naive height’ of a rational number. We say that a subset $S \subset \mathbb{Q}$ has density d if

$$\lim_{X \rightarrow \infty} \frac{\#\{a \in S \mid H(a) < X\}}{\#\{a \in \mathbb{Q} \mid H(a) < X\}} = d.$$

There is a folklore ‘minimalistic conjecture’:

Conjecture 8.2. *Let $E/\mathbb{Q}(t)$ be an elliptic curve of Mordell-Weil rank r . For $a \in \mathbb{Q}$ write E_a for its specialisation $t \mapsto a$. Then*

$$\text{rk } E_a/\mathbb{Q} = \begin{cases} r, & \text{if } w(E_a/\mathbb{Q}) = (-1)^r \\ r+1, & \text{otherwise} \end{cases}$$

for a set of rational numbers a of density 1.

Note that E_a is indeed an elliptic curve for all but finitely many a , so the conjecture makes sense. What it says is that generically, the rank of a fiber is a sum of the ‘geometric’ contribution from the points in the family plus an ‘arithmetic’ contribution from the root number.

8.1. Semistable curves in cubic extensions. The simplest elliptic curves are *semistable* ones, and their root numbers are particularly nice:

Definition 8.3. An elliptic curve over E/K is *semistable* if it has good or multiplicative reduction at all primes of K .

Because places of good and non-split multiplicative reduction do not contribute to the global root number, and infinite and split multiplicative places contribute -1 , for semistable E/K ,

$$w(E/K) = (-1)^{\#\{v|\infty\}} (-1)^{\#\{v \text{ split multiplicative for } E\}}.$$

Thus, the parity conjecture implies

Conjecture 8.4. *If E/\mathbb{Q} is semistable, then*

$$\text{rk } E/\mathbb{Q} \equiv 1 + \#\{\text{primes } p \text{ where } E \text{ has split mult. red.}\} \pmod{2}.$$

Example 8.5 ([18]). Take $E = 19A3$ over \mathbb{Q} ,

$$E : y^2 + y = x^3 + x^2 + x, \quad \Delta = 19, \text{ split mult. at } 19.$$

Two-descent shows that its rank over \mathbb{Q} is 0 (in accordance with the prediction that it is even).

Now take $K_m = \mathbb{Q}(\sqrt[3]{m})$ for some cube-free integer $m > 1$. Then the conjecture asserts that

$$\text{rk } E/K_m \equiv 2 + \begin{cases} 3, & 19 \text{ splits in } K_m \\ 1, & \text{otherwise} \end{cases} \pmod{2}.$$

$\# \{v|\infty\} \qquad \qquad \# \{v|19\}$

So $\text{rk } E/K_m$ should always be odd, in particular positive. We get the following surprising statement:

Conjecture 8.6. *For every $m > 1$, not a cube, the equation $y^2 + y = x^3 + x^2 + x$ has infinitely many solutions in $\mathbb{Q}(\sqrt[3]{m})$.*

It is known that an elliptic curve over \mathbb{Q} acquires rank over infinitely many fields of the form $\mathbb{Q}(\sqrt[3]{m})$ ([9] Thm. 1), but the full conjecture appears to be completely unapproachable at the moment.

8.2. Number fields K such that $w(E/K) = 1$ for all E/\mathbb{Q} .

Let $K = \mathbb{Q}(\sqrt{-1}, \sqrt{17})$. This field has a peculiar property that every place of \mathbb{Q} splits into an even number of places in it (2 or 4):

- K has 2 (complex) places $v|\infty$.
- 2 splits in $\mathbb{Q}(\sqrt{17})$, and thus in K as well.
- 17 splits in $\mathbb{Q}(\sqrt{-1})$, and thus in K as well.
- Primes $p \neq 2, 17$ are unramified in K/\mathbb{Q} , so their decomposition groups are cyclic, $D_p \neq \text{Gal}(K/\mathbb{Q}) = C_2 \times C_2$; so such p split as well.

Thus, for *any* elliptic curve E/\mathbb{Q} ,

$$w(E/K) = \prod_{\mathfrak{p}} w(E/K_{\mathfrak{p}}) = \prod_p (\pm 1)^{\text{even}} = +1,$$

and we get

Conjecture 8.7. *Every elliptic curve E/\mathbb{Q} has even rank over $\mathbb{Q}(\sqrt{-1}, \sqrt{17})$.*

The existence of such number fields was pointed out to us by Rubin. Note also that the same conjecture may be stated for abelian varieties. As an exercise, we leave it to the reader to use the same ideas to show that the parity conjecture implies the following:

Conjecture 8.8. *Every elliptic curve over \mathbb{Q} with split multiplicative reduction at 2 has infinitely many rational points over $\mathbb{Q}(\zeta_8)$.*

8.3. Goldfeld's conjecture over \mathbb{Q} .

Definition 8.9. For an elliptic curve

$$E/\mathbb{Q} : y^2 = f(x)$$

and a (usually square-free) integer d , the *quadratic twist* of E by d is

$$E_d/\mathbb{Q} : dy^2 = f(x).$$

Note that $E \cong E_d$ over $\mathbb{Q}(\sqrt{d})$, but not over \mathbb{Q} . Now, if $d_0 < 0$ is such that all primes $p|2\Delta_E$ split in $\mathbb{Q}(\sqrt{d_0})$, then it is easy to see that

$$w(E_{dd_0}/\mathbb{Q}) = -w(E_d/\mathbb{Q}) \quad \text{for all square-free } d.$$

In other words, the involution $d \leftrightarrow dd_0$ on $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ changes the sign of $w(E_d)$. So

$$\begin{aligned} w(E_d/\mathbb{Q}) &= +1 \text{ for 50\% square-free } d\text{'s}, \\ w(E_d/\mathbb{Q}) &= -1 \text{ for 50\% square-free } d\text{'s}, \end{aligned}$$

meaning that

$$(8.10) \quad \frac{\#\{ |d| \leq X \text{ square-free } |w(E_d/\mathbb{Q}) = 1 \}}{\#\{ |d| \leq X \text{ square-free } \}} \longrightarrow \frac{1}{2} \quad \text{as } X \rightarrow \infty.$$

The ‘minimalistic conjecture’ above becomes a famous conjecture of Goldfeld:

Conjecture 8.11 (Goldfeld). *Let E/\mathbb{Q} be an elliptic curve. Then*

$$\begin{aligned} \text{rk } E_d/\mathbb{Q} &= 0 \quad \text{for 50\% square-free } d\text{'s}, \\ \text{rk } E_d/\mathbb{Q} &= 1 \quad \text{for 50\% square-free } d\text{'s}, \\ \text{rk } E_d/\mathbb{Q} &\geq 2 \quad \text{for 0\% square-free } d\text{'s}. \end{aligned}$$

Note that ‘0%’ does not exclude the possibility of E having infinitely many quadratic twists of rank ≥ 2 . It only says that

$$r_{\geq 2}(X) := \#\{ |d| \leq X \text{ square-free } | \text{rk } E_d/\mathbb{Q} \geq 2 \}$$

is $o(X)$ (the denominator in (8.10) is $\sim X/\zeta(2)$ for large X). In fact, it is known that

$$\begin{aligned} r_{=0}(X) &\geq X/\log X && \text{(Ono-Skinner [44])} \\ r_{=1}(X) &\geq X^{1-\epsilon} && \text{(Pomykala-Perelli [45])} \\ r_{\geq 2}(X) &\geq C_E X^{1/7}/\log^2 X && \text{(Stewart-Top [56])}. \end{aligned}$$

For some specific elliptic curves it is known that $r_{=0}(X) \sim CX$ and $r_{=1}(X) \sim C'X$, but to get C or C' to be $\frac{1}{2}$ seems to be extremely hard.

8.4. No Goldfeld over number fields. Over number fields Goldfeld's Conjecture has to be formulated differently, because the ' $w = +1$ in 50% cases'-formula (8.10) may not hold. The simplest counterexample is CM curves:

Example 8.12. Let $K = \mathbb{Q}(i)$ and $E/K : y^2 = x^3 + x$. This is a curve with complex multiplication,

$$\text{End}_K E \cong \mathbb{Z}[i], \quad [i](x, y) = (-x, iy).$$

The set of rational points $E(K)$ is naturally a $\mathbb{Z}[i]$ -module, and so is $E(F)$ for any extension F/K . Because $E(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}(i)$ -vector space, it is even-dimensional over \mathbb{Q} , so $\text{rk } E/F$ is even for every $F \supset K$. Hence

$$\text{rk } E_d/K = \text{rk } E/K(\sqrt{d}) - \text{rk } E/K \equiv 0 \pmod{2}, \quad \text{all } d \in K^\times/K^{\times 2},$$

in other words every quadratic twist of E/K has even rank. It is also not hard to prove that $w(E_d/K) = 1$ for all d , as expected.

The same applies to any CM curve E/K with endomorphisms defined over K . Such a K is automatically totally complex (it contains $\text{End } K$ as a subring), the representation $G_K \rightarrow \text{Aut } T_l E$ has abelian image, and E acquires everywhere good reduction after an abelian extension of K . Interestingly, these are precisely the local conditions on an elliptic curve to guarantee that all of its quadratic twists have the same root number:

Theorem 8.13 ([10]). *Let E/K be an elliptic curve. Then $w(E_d/K) = w(E/K)$ for all $d \in K^\times$ if and only if*

- (a) *K is totally complex, and*
- (b) *For all primes \mathfrak{p} of K the curve E/K_v acquires good reduction after an abelian extension of K_v .*

For semistable curves the second condition simply says that E must have everywhere good reduction. It is not hard to construct explicit examples:

Example 8.14. The elliptic curve $E/\mathbb{Q} : y^2 = x^3 + \frac{5}{4}x^2 - 2x - 7$ (121C1) has minimal discriminant 11^4 and acquires everywhere good reduction over, e.g., $\mathbb{Q}(\sqrt[3]{11})$. If we take for instance $K = \mathbb{Q}(\zeta_3, \sqrt[3]{11})$, it is totally complex, E/K has everywhere good reduction, so

$$w(E_d/K) = w(E/K) = (-1)^{\#\{v|\infty\}} = -1, \quad \text{for all } d \in K^\times.$$

The parity conjecture implies that every quadratic twist of E/K has positive rank, or, equivalently, that the rank of E must grow in *every* quadratic extension of K .

Conjecture 8.15. *The curve 121C1 over $K = \mathbb{Q}(\zeta_3, \sqrt[3]{11})$ and all of its quadratic twists over K have positive rank.*

Here is a very elementary way to phrase this:

Conjecture 8.16. *Over $K = \mathbb{Q}(\zeta_3, \sqrt[3]{11})$ the polynomial $x^3 + \frac{5}{4}x^2 - 2x - 7 \in K[x]$ takes every value in $K^\times/K^{\times 2}$.*

8.5. No local expression for the rank. The reader may have noticed that the above examples rely not so much on the precise formulae for the root numbers but mostly just on their existence. In other words, they explore the fact that (conjecturally) the parity of the Mordell-Weil rank is a ‘sum of local invariants’:

Definition 8.17. Say that the Mordell-Weil rank (resp. Mordell-Weil rank modulo n) is a *sum of local invariants* if there is a \mathbb{Z} -valued function $(k, E) \mapsto \Lambda(E/k)$ of elliptic curves over local fields⁶, such that for any elliptic curve E over any number field K ,

$$\mathrm{rk} E/K = \sum_v \lambda(E/K_v) \quad (\text{resp. } \mathrm{rk} E/K \equiv \sum_v \lambda(E/K_v) \pmod{n}),$$

the sum taken over all places of K (and implicitly finite).

The parity conjecture implies that the Mordell-Weil rank modulo 2 is a sum of local invariants, namely those defined by the local root numbers,

$$(-1)^{\lambda(E/K_v)} := w(E/K_v).$$

One might ask whether there is a local expression like this for the rank modulo 3 or modulo 4, or even for the rank itself. The answer is ‘no’:

Theorem 8.18 ([17]). *The Mordell-Weil rank is not a sum of local invariants. In fact, a stronger statement holds: for $n \in \{3, 4, 5\}$ the Mordell-Weil rank modulo n is not a sum of local invariants.*

Proof. Take $E/\mathbb{Q} : y^2 = x(x+2)(x-3)$, which is 480a1 in Cremona’s notation. Writing ζ_p for a primitive p th root of unity, let

$$F_n = \begin{cases} \text{the degree 9 subfield of } \mathbb{Q}(\zeta_{13}, \zeta_{103}) & \text{if } n = 3, \\ \text{the degree 25 subfield of } \mathbb{Q}(\zeta_{11}, \zeta_{241}) & \text{if } n = 5, \\ \mathbb{Q}(\sqrt{-1}, \sqrt{41}, \sqrt{73}) & \text{if } n = 4. \end{cases}$$

Because 13 and 103 are cubes modulo one another, and all other primes are unramified in F_3 , every place of \mathbb{Q} splits into 3 or 9 in F_3 . Similarly in F_4 (resp. F_5) every place of \mathbb{Q} splits into a multiple of 4 (resp. 5) places. Hence, if the Mordell-Weil rank modulo n were a sum of local invariants, it would be $0 \in \mathbb{Z}/n\mathbb{Z}$ for E/F_n .

However, 2-descent shows that $\mathrm{rk} E/F_3 = \mathrm{rk} E/F_5 = 1$ and $\mathrm{rk} E/F_4 = 6$ (e.g. using Magma over all minimal non-trivial subfields of F_n). \square

Remark 8.19. It is interesting to note that the L -series of the curve $E = 480a1$ used in the proof over $F = F_4 = \mathbb{Q}(\sqrt{-1}, \sqrt{41}, \sqrt{73})$ is formally a 4th power, in the sense that each Euler factor is:

$$L(E/F, s) = 1 \cdot \left(\frac{1}{1-3^{-2s}}\right)^4 \left(\frac{1}{1-5^{-2s}}\right)^4 \left(\frac{1}{1+14 \cdot 7^{-2s} + 7^2 \cdot 4s}\right)^4 \left(\frac{1}{1+6 \cdot 11^{-2s} + 11^2 \cdot 4s}\right)^4 \dots$$

⁶Meaning that if $k \cong k'$ and E/k and E'/k' are isomorphic elliptic curves (identifying k with k'), then $\Lambda(E/k) = \Lambda(E'/k')$.

However, it is not a 4th power of an entire function, as it vanishes to order 6 at $s = 1$. (Actually, it is not even a square of an entire function: a computation shows it has a simple zero at $1 + 2.1565479...i$.)

In fact, by construction of F , for any E/\mathbb{Q} the L -series $L(E/F, s)$ is formally a 4th power and vanishes to even order at $s = 1$ by the functional equation. Its square root has analytic continuation to a domain including $\operatorname{Re} s > \frac{3}{2}$, $\operatorname{Re} s < \frac{1}{2}$ and the real axis, and satisfies a functional equation $s \leftrightarrow 2 - s$, but it is not clear whether it has any arithmetic meaning.

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